III. Bakerian Lecture, 1917.—*The Configurations of Rotating Compressible Masses.*

By J. H. Jeans, *M.A.*, *F.R.S.*

Received May 14,—Lecture delivered May 17, 1917.

1. On the supposition that astronomical matter may be treated as incompressible and homogeneous, a single star rotating freely in stable equilibrium can be spheroidal or ellipsoidal, but of no other shape, while Darwin has shown that both components of a binary star must be very approximately of the ellipsoidal shape.

Both for the interpretation of astronomical observations and for the more general purposes of cosmogony, it becomes of importance to examine how the sequence of figures assumed by an ideal homogeneous mass will be modified by the compressibility and non-homogeneity of actual astronomical matter.

The general mathematical problem of determining the configurations of stable equilibrium of the most general compressible mass is one of great complexity, but some important simplifications can be introduced by the use of general considerations. These are discussed in §§ 2–5 of the present paper. In § 10 we abandon the general problem and turn to a detailed study of the configurations possible when the compressibility is such that pressure and density are connected by the law

$$p = \kappa\rho^{\gamma} - \text{cons.},$$

where κ and γ are constants, this of course including the important case of a gas in convective equilibrium. The results obtained are summarised, and their astronomical bearings discussed, in §§ 45–58. The paper is arranged so that these last sections contain the main results of the paper in a form which is free from mathematical technicalities; it is hoped that they will prove intelligible to readers who have omitted the more mathematical sections.

2. For a mass of matter of the most general kind, rotating with angular velocity ω about the axis of z, the equations of relative equilibrium are three of the type

$$\frac{\partial p}{\partial x} = \rho\frac{\partial \Omega}{\partial x}, \text{ \&c.}, \quad \ldots \quad (1)$$

where

$$\Omega = \mathrm{V} + \tfrac{1}{2}\omega^2(x^2+y^2). \quad \ldots \quad (2)$$

Here V is the total gravitational potential, including that of tidal forces from neighbouring stars, if any are present. From equation (1) it follows that the surfaces $p =$ cons., $\Omega =$ cons., and $\rho =$ cons., all coincide, so that the free surface, being a surface of constant pressure, must also be a surface of constant density, say $\rho = \sigma$, and must also be one of the equipotentials $\Omega =$ cons., say $\Omega = C$. The value of the gravitational potential over the surface must accordingly be

$$V = C - \tfrac{1}{2}\omega^2 (x^2 + y^2),$$

this being the potential of the rotating mass itself and of certain tide-generating masses outside. We may suppose the positions and structure of these tide-generating masses to be given, so that $\nabla^2 V$ is given at all points outside the surface of the rotating mass, while V is given at all points on the surface by the above equation, and vanishes at infinity. By a fundamental theorem in potential theory, it follows that V is determined uniquely at every point outside the mass in terms of ω, C and the shape of the boundary, whence $\partial V/\partial n$ must be determined uniquely in terms of these quantities at every point of the boundary.

If M is the total mass of the rotating body,

$$4\pi M = \iint \frac{\partial V}{\partial n}\, dS,$$

where the integral is taken over the surface of the body, and on substituting the value of $\partial V/\partial n$, this becomes a linear equation, which may be regarded as determining C uniquely in terms of M.

Thus when M, ω and the equation of the boundary are given, it appears that V and $\partial V/\partial n$ are uniquely determined over the boundary. If the value of ρ at the boundary, say σ, is given, and if we also know the law of compressibility at the boundary, then ρ and $d\rho/dn$ are uniquely determined over the boundary. Thus not one, but two, surfaces of constant density are fixed, namely the boundary S and the surface just inside it, say S′.

The mass, say M′, inside S′ is in equilibrium under the rotation ω and gravitational forces which originate from the external tide-generating masses and also from the layer of matter between S and S′. It now follows from the preceding argument that the surface of constant density next inside S′, say S″, is also determined. There are now three surfaces of constant density fixed, and by a continual repetition of the foregoing process, we can fix all such surfaces in turn.* Thus when the external boundary is given, and also M, ω and the tide-generating masses, which may be

* This cannot be defended as a piece of rigorous mathematical reasoning, but there can, I think, be no doubt that it is true for practical purposes. I have discussed the mathematical complications elsewhere ('Roy. Soc. Proc.,' A, vol. 98, p. 413). A more formal proof of the theorem will be found in the 'Monthly Notices of the R.A.S.,' vol. 77, p. 187.

regarded as the data of the problem, the interior arrangement of the matter is uniquely determined.

3. The first consequence of the foregoing theorem is that when the boundary of a mass is fixed, all the internal vibrations are necessarily stable. For the change from stability to instability can only occur through a vibration of zero-frequency, and this would require that there should be two contiguous equilibrium arrangements of the interior matter, a possibility which is excluded by the result just obtained.

The only possible configuration for a mass at rest and under no tidal forces will clearly be one in which the boundary is spherical and the surfaces of constant density are also spherical and concentric with the boundary.* If the mass is set into slow rotation, this system of concentric spheres will give place to a system of concentric spheroids. As the rotation increases further, the surfaces of equal density will no longer be strictly spheroidal, but it is clear that there must always be a linear series of configurations of equilibrium in which the boundary has the shape of a figure of revolution. This series of course reduces to the series of Maclaurin spheroids when the matter is incompressible.

Excluding for the present the case in which σ (the density at the boundary), vanishes, it can be shown† that there must be an infinite number of points of bifurcation on this series of figures of revolution, these corresponding to the different sectorial harmonics of the figure. The first point of bifurcation corresponds to the second sectorial harmonic: when this is reached the circular cross-section gives place to a slightly elliptic cross-section, and this leads to a series of figures having three planes of symmetry and three unequal axes, these figures reducing to the Jacobian ellipsoids when the matter is homogeneous and incompressible. It is convenient to refer to these two series as the series of pseudo-spheroids and pseudo-ellipsoids respectively. The first point of bifurcation on the series of pseudo-ellipsoids corresponds to a third harmonic displacement and leads to a series of pear-shaped figures.

It seems almost certain, although it has not been rigorously proved, that this last series of figures ends by fission into two detached masses revolving round one another as in the ordinary binary star formation. Assuming this, we may regard the passing of the first point of bifurcation on the series of pseudo-ellipsoids as the beginning of the process of fission. Until this stage is reached we have seen that the only possible figures of equilibrium for a rotating mass are pseudo-spheroids and pseudo-ellipsoids. For an incompressible mass the pear-shaped figures are unstable, so that spheroids and ellipsoids are the only possible figures of stable equilibrium. If the pear-shaped

* The special application of this to the figure of the Earth has been discussed in a separate paper ('Roy. Soc. Proc.,' A, vol. 98, p. 413).

† 'Monthly Notices of the R.A.S.,' vol. 77, p. 189. In this paper I was mistaken in thinking that $\sigma = 0$ presents a true exception to the general theory. The general argument there given failed to prove the result in the special case of $\sigma = 0$, but the result is true nevertheless, as will appear from the present paper.

figure proves to be unstable also for all compressible masses, then the only possible figures of stable equilibrium for a compressible mass will be pseudo-spheroids and pseudo-ellipsoids.

4. When the mass is compressible a complication can occur which, as it happens, does not arise in the incompressible problem. The resultant normal force at any point on the surface of rotating mass is $-\partial\Omega/\partial n$, this being the resultant of gravity and centrifugal force. In the incompressible problem, $\partial\Omega/\partial n$ does not vanish, except in the unstable configurations at the far ends of the spheroidal and ellipsoidal series, but this is not necessarily the case in the compressible problem. We shall find that $\partial\Omega/\partial n$ can vanish either on the series of pseudo-spheroids or on the series of pseudo-ellipsoids, and when this happens matter will necessarily be thrown off at the points at which $\partial\Omega/\partial n$ vanishes. The series of figures of equilibrium may accordingly be abruptly terminated at any stage by the vanishing of $\partial\Omega/\partial n$.

5. Thus it appears that the series of figures of equilibrium for a compressible mass, until the stage at which fission begins, will consist of pseudo-spheroids and pseudo-ellipsoids, these series possibly being abruptly terminated by the vanishing of $\partial\Omega/\partial n$ at any point. Our problem is to study these series of configurations; our method will be as follows:—

For an incompressible mass the density at the centre, which we shall denote by ρ_0, is identical with the density at the boundary, which we denote by σ. For a compressible mass, ρ_0 will be different from σ, and a rough measure of the extent to which the density differs from uniformity will be given by a quantity ϵ defined by

$$\epsilon = \frac{\rho_0 - \sigma}{\rho_0} \quad \text{. . . . .} \quad (3)$$

We know the solution of the problem when $\epsilon = 0$; we require to obtain it for all values of ϵ. Our method is to adopt the known solution when $\epsilon = 0$ as generating solution and to obtain, by what amounts to a method of successive approximations, an expansion for Ω in powers of ϵ. The success of the method will depend on the extent to which the series so obtained is convergent.

The value of Ω obtained in this way will be a function of x, y, z, ϵ and of the constants which enter into the law of compressibility. When x, y, z are small the value of Ω will be found to be convergent for all values of ϵ, but as x, y, z increase, the range of convergence of the series contracts. But in the most important case we shall consider, it is found that the series is convergent, even for points furthest removed from the origin, for values of ϵ up to some value between two and three. The largest value of ϵ which is of physical interest is $\epsilon = 1$, corresponding to $\sigma = 0$, and for this value of ϵ the series converges with considerable rapidity, so that the first few terms will give a fair approximation to the truth.

Our method, then, is to obtain the series of pseudo-spheroids and pseudo-ellipsoids as deformations of the already known series of spheroids and ellipsoids, expanding in

powers of the parameter ϵ. The boundaries may accordingly be regarded as distorted ellipsoids.

In a previous paper* I showed how to obtain the potential of a homogeneous mass, expanded in powers of a parameter which measured its divergence from the ellipsoidal shape. As a preliminary to the present investigation, we must examine the corresponding problem when the mass is not homogeneous.

Potential of a Non-homogeneous Distorted Ellipsoid.

6. We shall assume, as being adequate for the present problem, that it is possible to expand the density in the form

$$\rho = \rho_0 - \rho_2 - \rho_3 - \rho_4 - \ldots, \qquad (4)$$

where ρ_0 is the density at the origin, which is taken to be the point of maximum density, and ρ_2, ρ_3, ρ_4, ..., are functions of x, y, z of degrees 2, 3, 4 ..., respectively. No terms of degree unity occur because, from our choice of origin, the first differential coefficients of ρ vanish at the origin. The value of ρ_2 is

$$\rho_2 = -\tfrac{1}{2}\left(x^2 \frac{\partial^2 \rho}{\partial x^2} + 2xy \frac{\partial^2 \rho}{\partial x \, \partial y} + \ldots\right),$$

the differential coefficients all being evaluated at the origin. Our choice of origin has been such that ρ_2 is necessarily positive for all values of x, y and z, so that by a suitable choice of direction of axes, it must be possible to express ρ_2 in the form

$$\rho_2 = (\rho_0 - \sigma)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right).$$

Let us further put

$$\rho_3 + \rho_4 + \ldots = (\rho_0 - \sigma)\,\epsilon \mathrm{P}_0,$$

where ϵ is a numerical quantity which may for the present be left undefined, but will ultimately be taken to be the parameter defined by equation (3), and P_0 is a function of x, y and z. Then the general value of ρ, as given by equation (4), becomes

$$\rho = \rho_0 - (\rho_0 - \sigma)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \epsilon \mathrm{P}_0\right), \qquad (5)$$

and the boundary, defined by $\rho = \sigma$, has for its equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \epsilon \mathrm{P}_0 = 1. \qquad (6)$$

When ϵ is not too great this is a distorted ellipsoid. In the paper already referred to, I showed how to write down the potential of a homogeneous mass whose

* 'Phil. Trans.,' A, vol. 215, p. 27.

boundary was given by an equation of the type of (6), the solution occurring as a series of powers of ϵ. It is now necessary to extend the method so as to be able to write down the potential when the internal density is variable and given by equation (5).

As a matter of convenience, of which the advantage will appear later, we shall permit P_0 to include terms of degree 2 as well as terms of degree 3 and higher. This is equivalent to regarding the mass under discussion as being arrived at by distortion from an ellipsoid of density,

$$\rho = \rho_0-(\rho_0-\sigma)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

the only restriction on the distortion being that the point of maximum density must remain the origin, and that the density at this point must remain equal to ρ_0. But distortions which are themselves ellipsoidal are not prohibited, as they would be if second-degree terms were excluded from P_0.

7. Let q be a function of the density ρ, defined by

$$q^2 = \frac{\rho_0-\rho}{\rho_0-\sigma}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

so that, as we pass from the centre to the boundary, q varies continuously from 0 to 1. The surface of constant density ρ has for its equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+\epsilon P_0 = q^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

which may be regarded as a distorted ellipsoid of semi-axes qa, qb, qc. The boundary now figures as the special case of $q = 1$.

Let the potential of a homogeneous mass of unit density bounded by the surface (9) be denoted by $V_0(q)$ when evaluated at a point outside the surface, and by $V_i(q)$ when evaluated at a point inside the surface.

The whole heterogeneous mass of density given by equation (5) may be regarded as split up into shells of constant density, and the potential of the whole will be found to be given by the formulæ:

$$V_0 = \sigma V_0(1)+\int_\sigma^{\rho_0} V_0(q)\, d\rho \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

$$V_i = \sigma V_i(1)+\int_\sigma^{\rho} V_i(q)\, d\rho+\int_\rho^{\rho_0} V_0(q)\, d\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

the first formula giving the potential at a point entirely outside the mass, and the second formula giving the potential at a point inside the mass at which the density is ρ.

8. Suppose that P_0 is expressed in the form

$$P_0 = F\left(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}\right)$$

and let P be a function of x, y, z and a new variable μ, given by

$$P = F\left(\frac{q^2x}{q^2a^2+\mu}, \frac{q^2y}{q^2b^2+\mu}, \frac{q^2z}{q^2c^2+\mu}\right)$$

so that P_0 is the value of P when $\mu = 0$. Further introduce ξ', η', ζ' defined by

$$\xi' = x/(q^2a^2+\mu), \text{ \&c.},$$

so that

$$P = F(q^2\xi', q^2\eta', q^2\zeta')$$

and f and D (an operator) defined by

$$f = \frac{x^2}{q^2a^2+\mu} + \frac{y^2}{q^2b^2+\mu} + \frac{z^2}{q^2c^2+\mu} - 1 = (q^2a^2+\mu)\,\xi'^2 + \ldots - 1,$$

$$D = \left(\frac{1}{q^2a^2} - \frac{1}{q^2a^2+\mu}\right)\frac{\partial^2}{\partial\xi'^2} + \left(\frac{1}{q^2b^2} - \frac{1}{q^2b^2+\mu}\right)\frac{\partial^2}{\partial\eta'^2} + \left(\frac{1}{q^2c^2} - \frac{1}{q^2c^2+\mu}\right)\frac{\partial^2}{\partial\zeta'^2}.$$

Let $\phi(q)$, a function of ξ', η', ζ', μ and q, be given by

$$\begin{aligned}\phi(q) = \epsilon\left[P - \frac{1}{4}fDP + \frac{1}{2^2}\left(\frac{1}{4}f\right)^2 D^2P - \frac{1}{2^2 . 3^2}\left(\frac{1}{4}f\right)^3 D^3P + \ldots\right] \\ - \frac{1}{8}\epsilon^2\left[DP^2 - \frac{1}{8}fD^2P^2 + \frac{1}{192}f^2D^3P^2 - \frac{1}{9216}f^3D^4P^2 + \ldots\right] \\ + \frac{1}{192}\epsilon^3[D^2P^3 - \ldots], \text{ \&c.} \qquad (12)\end{aligned}$$

When $\mu = 0$, $D = 0$ and $P = P_0$, so that $\phi(q)$ reduces to ϵP_0, and

$$f + \frac{\phi(q)}{q^2} = \frac{1}{q^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + \epsilon P_0 - q^2\right).$$

Thus the surface of constant density ρ is the surface $\mu = 0$ in the family of surfaces

$$f + \phi(q)/q^2 = 0. \qquad (13)$$

From the previous investigation* it now follows that the value of $V_0(q)$ is

$$V_0(q) = -\pi\int_{\mu'}^{\infty}\left(f+\frac{\phi(q)}{q^2}\right)\frac{q^3abc\,d\mu}{[(q^2a^2+\mu)(q^2b^2+\mu)(q^2c^2+\mu)]^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad (14)$$

in which the lower limit of integration μ' is the root of equation (13) at the external point x, y, z at which the potential is being evaluated. The same formula (14) with μ' put equal to zero will give the value of $V_i(q)$, the potential at an internal point.

To transform these expressions into a form suitable for use in the present problem, introduce new variables $\lambda, \xi, \eta, \zeta$ to replace μ, ξ', η', ζ', these being defined by

$$\lambda = \mu/q^2, \qquad \xi = q^2\xi' = x/(a^2+\lambda), \text{ \&c.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

We now have

$$P = F(\xi, \eta, \zeta),$$

$$q^2f = \frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}-q^2,$$

$$\frac{D}{q^2} = \left(\frac{1}{a^2}-\frac{1}{a^2+\lambda}\right)\frac{\partial^2}{\partial\xi^2}+\left(\frac{1}{b^2}-\frac{1}{b^2+\lambda}\right)\frac{\partial^2}{\partial\eta^2}+\left(\frac{1}{c^2}-\frac{1}{c^2+\lambda}\right)\frac{\partial^2}{\partial\zeta^2},$$

while formula (14) becomes

$$V_0(q) = -\pi\int_{\lambda'}^{\infty}[q^2f+\phi(q)]\frac{abc\,d\lambda}{\Delta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

in which Δ stands for $[(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)]^{\frac{1}{2}}$, and λ' is the root of

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}+\phi(q) = q^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

The same formula (16) in which λ' is put equal to zero will of course give the value of $V_i(q)$.

9. Attacking equations (10) and (11), we now have

$$\int_{\sigma}^{\rho_0}V_0(q)\,d\rho = (\rho_0-\sigma)\int_0^1 V_0(q)\,dq^2$$

$$= (\rho_0-\sigma)\left[V_0(1)-\int_0^1 q^2\frac{dV_0(q)}{dq^2}\,dq^2\right],$$

so that equation (10) becomes

$$V_0 = \rho_0V_0(1)-(\rho_0-\sigma)E_0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

* 'Phil. Trans.,' A, vol. 215, p. 27.

where

$$E_0 = \int_0^1 q^2 \frac{dV_0(q)}{dq^2}\, dq^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

This expresses that V_0 is equal to the potential of a homogeneous solid of density ρ_0 *minus* $(\rho_0 - \sigma)$ times the potential E_0, this last term accordingly representing the effect of the falling off of density from the maximum value ρ_0.

Similarly equation (11) becomes

$$V_i = \rho_0 V_i(1) - (\rho_0 - \sigma) E_i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where

$$E_i = \int_0^q q^2 \frac{dV_0(q)}{dq^2}\, dq^2 + \int_q^1 q^2 \frac{dV_i(q)}{dq^2}\, dq^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

in which the limit of integration is the value of q at the internal point at which the potential is being evaluated.

The value of $V_0(q)$, as given by formula (16), is a function of q^2 and λ', the two being connected by equation (17). Thus we have

$$\frac{dV_0(q)}{dq^2} = \frac{\partial V_0(q)}{\partial q^2} + \frac{\partial V_0(q)}{\partial \lambda'} \frac{\partial \lambda'}{\partial q^2},$$

and the last term vanishes since $\partial V_0(q)/\partial \lambda'$ contains $[q^2 f + \phi(q)]$ as a factor. Thus we have

$$\frac{dV_0(q)}{dq^2} = \frac{\partial V_0(q)}{\partial q^2} = \pi abc \int_{\lambda'}^{\infty} \left(1 - \frac{\partial \phi(q)}{\partial q^2}\right) \frac{\partial \lambda}{\Delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

and the value of $dV_i(q)/dq^2$ is given by the same formula with λ' put equal to zero.

Using these values we find

$$E_0 = \pi abc \int_0^1 q^2 \left[\int_{\lambda'}^{\infty} \left(1 - \frac{\partial \phi(q)}{\partial q^2}\right) \frac{d\lambda}{\Delta}\right] dq^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$E_i = \pi abc \int_0^q q^2 \left[\int_{\lambda'}^{\infty} \left(1 - \frac{\partial \phi(q)}{\partial q^2}\right) \frac{d\lambda}{\Delta}\right] dq^2 + \pi abc \int_q^{\infty} q^2 \left[\int_0^{\infty} \left(1 - \frac{\partial \phi(q)}{\partial q^2}\right) \frac{d\lambda}{\Delta}\right] dq^2. \quad . \quad (24)$$

Both values of E may be regarded as given by a double integration in a plane in which λ and q are rectangular co-ordinates. Let us first consider E_0.

In fig. 1, let PQ represent the curve whose equation is

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} + \phi(q) = q^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

Clearly, when $q = 0$, the value of λ is ∞, while when $q = 1$, λ has some value λ'', which is the root of

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} + \phi(1) = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26)$$

The value of E_0 is obtained by integrating over the area shaded in fig. 1. Changing the order of integration, we find

$$E_0 = \pi abc \int_{\lambda''}^{\infty} \left[\int_q^1 \left(1 - \frac{\partial \phi(q)}{\partial q^2} \right) dq^2 \right] \frac{d\lambda}{\Delta}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (27)$$

in which the lower limit q is the root of equation (25), while the lower limit λ'' is the root of equation (26).

The value of E_i is obtained by a double integration in the same plane over an area such as that shaded in fig. 2, the different directions of shading distinguishing the

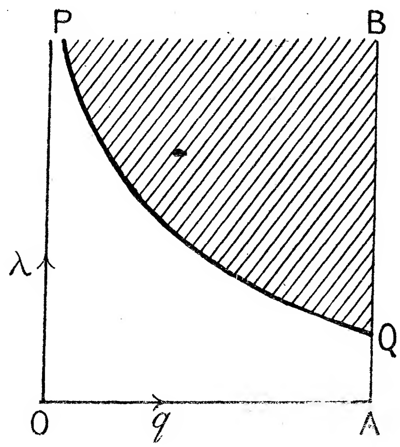

Fig. 1.

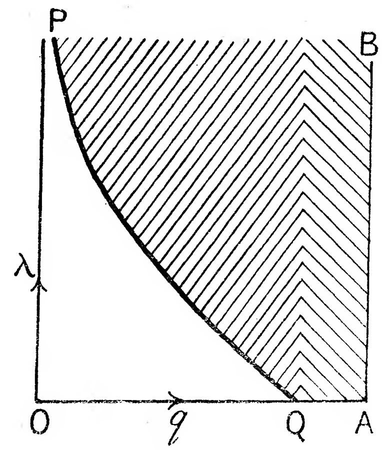

Fig. 2.

areas covered by the two separate integrals in equation (24). Again changing the order of integration, we obtain

$$E_i = \pi abc \int_0^{\infty} \left[\int_q^1 q^2 \left(1 - \frac{\partial \phi(q)}{\partial q^2} \right) dq^2 \right] \frac{d\lambda}{\Delta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (28)$$

in which the lower limit q is again the root of equation (25).

This completes the solution of the potential problem; we now attack the main problem of determining configurations of equilibrium.

General Equations of Equilibrium.

10. We can only find configurations of equilibrium by assuming a definite law to connect pressure with density. We shall accordingly assume the relation

$$p = \kappa\rho^{\gamma} - \text{cons.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (29)$$

The equations of equilibrium (1) now assume the form

$$\kappa\gamma\rho^{\gamma-2}\frac{\partial\rho}{\partial x} = \frac{\partial\Omega}{\partial x}, \text{ \&c.,}$$

and these have the common integral

$$\frac{\kappa\gamma}{\gamma-1}\rho^{\gamma-1} = \Omega+\text{cons.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

We have already (equations (3) and (5)) assumed the value of ρ to be

$$\rho = \rho_0\left[1-\epsilon\Sigma\frac{x^2}{a^2}-\epsilon^2\mathrm{P}_0\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

and on expanding $\rho^{\gamma-1}$ by the binomial theorem, equation (30) becomes

$$\Omega = \text{cons.}+\kappa\gamma\rho_0{}^{\gamma-1}\left[-\left(\epsilon\Sigma\frac{x^2}{a^2}+\epsilon^2\mathrm{P}_0\right)+\tfrac{1}{2}(\gamma-2)\left(\epsilon\Sigma\frac{x^2}{a^2}+\epsilon^2\mathrm{P}_0\right)^2\right.$$
$$\left.-\tfrac{1}{6}(\gamma-2)(\gamma-3)\left(\epsilon\Sigma\frac{x^2}{a^2}+\epsilon^2\mathrm{P}_0\right)^3+\ldots\right]. \quad . \quad . \quad . \quad (32)$$

Let the figure of density given by equation (31) be supposed to be a figure of equilibrium under a rotation ω and tidal forces of potential V_T where

$$\mathrm{V}_\mathrm{T} = \tau_1x^2+\tau_2y^2+\tau_3z^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

in which of course $\tau_1+\tau_2+\tau_3 = 0$. We then have at any point inside the mass

$$\Omega = \mathrm{V}_i+\mathrm{V}_\mathrm{T}+\tfrac{1}{2}\omega^2(x^2+y^2)$$
$$= \rho_0[\mathrm{V}_i(1)-\epsilon\mathrm{E}_i]+(\tau_1x^2+\tau_2y^2+\tau_3z^2)+\tfrac{1}{2}\omega^2(x^2+y^2), \quad . \quad . \quad . \quad . \quad (34)$$

and on equating the right-hand members of equations (32) and (34) we obtain a solution of the problem.

11. The simplest solution occurs when the variations of density are small, so that ϵ is small; in this case $\kappa\gamma$ is large. We must not simply put $\epsilon = 0$ in equation (32), for otherwise Ω would reduce to a pure constant. We accordingly suppose $\kappa\gamma\epsilon$ to remain finite, and put $\epsilon = 0$ in all remaining terms. Thus the equation to be solved is

$$\rho_0\mathrm{V}_i(1)+(\tau_1x^2+\tau_2y^2+\tau_3z^2)+\tfrac{1}{2}\omega^2(x^2+y^2) = \text{cons.}-\kappa\gamma\epsilon\rho_0{}^{\gamma-1}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right). \quad . \quad (35)$$

The boundary reduces to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

and the value of $V_i(1)$, which is the potential of this ellipsoid filled with matter of unit density, is

$$V_i(1) = -\pi abc\,(J_A x^2+J_B y^2+J_C z^2-J), \quad \ldots \ldots \quad (36)$$

where the notation is that I have previously used* in which

$$J_{AB\ldots} = \int_0^\infty \frac{d\lambda}{\Delta\,(a^2+\lambda)\,(b^2+\lambda)\ldots} \quad \ldots \ldots \quad (37)$$

On substituting this value for $V_i(1)$ into equation (35) and equating coefficients of x^2, y^2, z^2, we obtain, as the conditions for equilibrium,

$$J_A-\frac{\tau_1}{\pi\rho_0 abc}-\frac{\omega^2}{2\pi\rho_0 abc}=\frac{\theta}{a^2}, \quad \ldots \ldots \quad (38)$$

$$J_B-\frac{\tau_2}{\pi\rho_0 abc}-\frac{\omega^2}{2\pi\rho_0 abc}=\frac{\theta}{b^2}, \quad \ldots \ldots \quad (39)$$

$$J_C-\frac{\tau_3}{\pi\rho_0 abc}=\frac{\theta}{c^2}, \quad \ldots \ldots \quad (40)$$

in which

$$\theta = \frac{\kappa\gamma\epsilon\rho_0^{\gamma-2}}{\pi abc}. \quad \ldots \ldots \quad (41)$$

By addition of the corresponding sides of the three equations (36) to (38), we obtain

$$\frac{2}{abc}-\frac{\omega^2}{\pi\rho_0 abc}=\theta\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right). \quad \ldots \ldots \quad (42)$$

It is now clear that equations (38) to (40) together with (41) are simply the equations which determine a, b, c, the semi-axes of the ellipsoid which is a figure of equilibrium for an incompressible mass. We have, however, found that as far as this first approximation the value of ρ is not necessarily constant throughout the ellipsoid; it is given by

$$\rho = \rho_0-(\rho_0-\sigma)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right). \quad \ldots \ldots \quad (43)$$

We have further found the relation connecting θ with the constants κ and γ. Substituting the value for θ from equation (41) into equation (32), and equating the

* 'Phil. Trans.,' A, vol. 215, p. 50.

value of Ω so obtained to the value given by equation (34), the general equation of equilibrium is obtained in the form

$$\rho_0\left[\mathrm{V}_i(1)-\epsilon\mathrm{E}_i\right]+(\tau_1x^2+\tau_2y^2+\tau_3z^2)+\tfrac{1}{2}\omega^2(x^2+y^2)$$
$$=\text{cons.}-\pi\rho_0abc\theta\left[\left(\Sigma\frac{x^2}{a^2}+\epsilon\mathrm{P}_0\right)-\tfrac{1}{2}\epsilon(\gamma-2)\left(\Sigma\frac{x^2}{a^2}+\epsilon\mathrm{P}_0\right)^2\right.$$
$$\left.+\tfrac{1}{6}\epsilon^2(\gamma-2)(\gamma-3)\left(\Sigma\frac{x^2}{a^2}+\epsilon\mathrm{P}_0\right)^3-\ldots\right]. \quad . \quad (44)$$

12. The first approximation has been obtained by putting $\epsilon=0$ in this equation; we now proceed to higher approximations. A second approximation will be obtained by omitting all power of ϵ beyond the first; a third approximation by not going beyond ϵ^2, and so on.

On replacing $\epsilon\mathrm{P}_0$ by $\epsilon\mathrm{P}_0+\epsilon^2\mathrm{Q}_0+\epsilon^3\mathrm{R}_0+\ldots$, we may suppose that the density expanded in powers of ϵ is

$$\rho=\rho_0\left[1-\epsilon\left(\Sigma\frac{x^2}{a^2}+\epsilon\mathrm{P}_0+\epsilon^2\mathrm{Q}_0+\epsilon^3\mathrm{R}_0+\ldots\right)\right], \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

so that the boundary, $\rho=\sigma$, is

$$\Sigma\frac{x^2}{a^2}+\epsilon\mathrm{P}_0+\epsilon^2\mathrm{Q}_0+\epsilon^3\mathrm{R}_0+\ldots=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

The general equation of equilibrium will be obtained from equation (44) on replacing $\epsilon\mathrm{P}_0$ by

$$\epsilon\mathrm{P}_0+\epsilon^2\mathrm{Q}_0+\epsilon^3\mathrm{R}_0+\ldots. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

The value of $\mathrm{V}_i(1)$ to be used in equation (42) will no longer be that given by equation (36); let us suppose the whole value to be

$$\mathrm{V}_i(1)+\epsilon\Delta\mathrm{V}_i(1)+\epsilon^2\delta\mathrm{V}_i(1)+\ldots, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

this being the internal potential of a homogeneous solid of unit density whose boundary is determined by equation (46). Similarly, let the whole value of E_i be supposed expanded in the form

$$\mathrm{E}_i+\epsilon\Delta\mathrm{E}_i+\epsilon^2\,\delta\mathrm{E}_i+\ldots.$$

Finally, let the value of ω^2 in the complete solution be supposed given by

$$\frac{\omega^2}{2\pi\rho_0abc}=n+\epsilon\,\Delta n+\epsilon^2\,\delta n+\ldots. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$

Then the complete equation (44) becomes, on dividing throughout by $\pi\rho_0 abc$,

$$\frac{1}{\pi abc}\left[V_i(1)+\epsilon\,\Delta V_i(1)+\epsilon^2\,\delta V_i(1)+\ldots-\epsilon\,(E_i+\epsilon\,\Delta E_i+\ldots)\right]$$

$$+\frac{\tau_1 x^2+\tau_2 y^2+\tau_3 z^2}{\pi\rho_0 abc}+(n+\epsilon\,\Delta n+\epsilon^2\,\delta n+\ldots)(x^2+y^2)$$

$$=\text{cons.}-\theta\left[\left(\Sigma\frac{x^2}{a^2}+\epsilon P_0+\epsilon^2 Q_0+\epsilon^3 R_0+\ldots\right)-\tfrac{1}{2}\epsilon\,(\gamma-2)\left(\Sigma\frac{x^2}{a^2}+\epsilon P_0+\epsilon^2 Q_0+\ldots\right)^2\right.$$

$$\left.+\tfrac{1}{6}\epsilon^2(\gamma-2)(\gamma-3)\left(\Sigma\frac{x^2}{a^2}+\epsilon P_0+\ldots\right)^3-\ldots\right], \quad . \quad . \quad . \quad (50)$$

an equation in which each side is equal to $\Omega/\pi\rho_0 abc$.

13. On equating terms which are independent of ϵ, we of course arrive merely at equations (38) to (40); these determine a, b, c when the data of the problem are given.

On equating terms in ϵ, we obtain

$$\frac{1}{\pi abc}(\Delta V_i(1)-E_i)+\Delta n\,(x^2+y^2)=-\theta\left[P_0-\tfrac{1}{2}(\gamma-2)\left(\Sigma\frac{x^2}{a^2}\right)^2\right]. \quad . \quad . \quad (51)$$

In evaluating E_i we put $\epsilon=0$, and so may neglect $\phi(q)$. Thus, from equation (28),

$$E_i=\pi abc\int_0^\infty \tfrac{1}{2}(1-q^4)\frac{d\lambda}{\Delta}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

in which, from equation (25), q is given by

$$q^2=\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

Thus, in the notation defined by equation (37),

$$E_i=-\tfrac{1}{2}\pi abc\,(J_{AA}x^4+J_{BB}y^4+J_{CC}z^4+2J_{BC}y^2z^2+\ldots-J). \quad . \quad . \quad . \quad . \quad (54)$$

Using this value for E_i, it appears that all terms in equation (51) are of degrees 4 and 2 except P_0 and $\Delta V_i(1)$. The value of $\Delta V_i(1)$ depends on that of P_0, and terms of degree n in P_0 give rise to terms of degrees n, $n-2$, ... in $\Delta V_i(1)$. It is accordingly clear that the solution of equation (51) must be such that P_0 consists of terms of degrees 4 and 2; the value of $\Delta V_i(1)$ is then also of degrees 4 and 2, and, on equating coefficients, we can determine the coefficients in P_0.

Following closely the notation previously used,* let us assume for P_0 the value

$$P_0=\frac{Lx^4}{a^8}+\frac{My^4}{b^8}+\frac{Nz^4}{c^8}+\frac{2ly^2z^2}{b^4c^4}+\frac{2mz^2x^2}{c^4a^4}+\frac{2nx^2y^2}{a^4b^4}+\frac{2px^2}{a^4}+\frac{2qy^2}{b^4}+\frac{2rz^2}{c^4}, \quad . \quad (55)$$

* 'Phil. Trans.,' A, vol. 215, p. 54. The small quantity $\frac{1}{2}e^2$ of that paper is replaced by ϵ in this paper, otherwise the figure of that paper reduces to the present figure on taking α, β, γ and s all zero.

so that (*cf.* § 8)

$$P = L\xi^4+M\eta^4+N\zeta^4+2l\eta^2\zeta^2+2m\zeta^2\xi^2+2n\xi^2\eta^2+2p\xi^2+2q\eta^2+2r\zeta^2. \quad . \quad . \quad (56)$$

As far as terms of first degree in ϵ, we may, from equation (12), take

$$\phi(q) = \epsilon[P-\tfrac{1}{4}fDP+\tfrac{1}{64}f^2D^2P], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

the remaining terms disappearing because P contains no terms of degree higher than four in ξ, η, ζ. The whole potential of the solid of unit density, as far as terms in ϵ, is from formula (14)

$$V_i(1)+\epsilon\Delta V_i(1) = -\pi abc\int_0^\infty[f+\phi(1)]\frac{d\lambda}{\Delta},$$

so that

$$\Delta V_i(1) = -\pi abc\int_0^\infty[P-\tfrac{1}{4}fDP+\tfrac{1}{64}f^2D^2P]\frac{d\lambda}{\Delta}, \quad . \quad . \quad . \quad . \quad (58)$$

in which q is put equal to 1.

Equation (51) now becomes

$$\int_0^\infty[P-\tfrac{1}{4}fDP+\tfrac{1}{64}f^2D^2P]\frac{d\lambda}{\Delta}-\theta P_0$$

$$= \tfrac{1}{2}(\Sigma J_{AA}x^4+2\Sigma J_{BC}y^2z^2-J)-\tfrac{1}{2}(\gamma-2)\theta\left(\Sigma\frac{x^2}{a^2}\right)^2+\Delta n(x^2+y^2). \quad . \quad . \quad (59)$$

Clearly the left-hand member of this equation will be a linear function of L, M, N, ..., while the right-hand member does not involve these coefficients. Thus the values of L, M, N, ... will each be the sum of a number of contributions corresponding to the different terms on the right of the equation. Let us suppose that

$$P_0 = P'_0+P''_0(\gamma-2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

and that

$$L = L'+L''(\gamma-2), \&c., \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

thus separating the contribution from the term in $(\gamma-2)$ from the remainder.

14. It will be remembered that $\epsilon\Delta V_i$ (1) is the increase of internal potential resulting from deforming the surface of an ellipsoid of unit density so that its surface is changed from

$$x^2/a^2+y^2/b^2+z^2/c^2-1 = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

into

$$x^2/a^2+y^2/b^2+z^2/c^2-1+\epsilon P_0 = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

Consider the special deformation in which P_0 is of the form

$$P_0 = \psi\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

where ψ is any function whatever. Equation (63) can be solved in the form

$$x^2/a^2+y^2/b^2+z^2/c^2-1-\eta = 0, \qquad (65)$$

and on comparison with equation (62) the deformation is seen to consist of the addition of a thin homeoidal shell of uniform density. By a well-known theorem, the potential of such a shell is constant at all internal points, so that for the particular deformation represented by equation (64), we have $\Delta V_i(1) =$ a constant. Incidentally, we may note that $\delta V_1(1)$ is also constant, and so on to all orders.

15. Returning to equation (59), it appears that P''_0 must satisfy

$$\int_0^\infty [P''-\tfrac{1}{4}f DP''+\tfrac{1}{64}f^2D^2P'']\frac{d\lambda}{\Delta}-\theta P_0'' = -\tfrac{1}{2}\theta\left(\Sigma\frac{x^2}{a^2}\right)^2$$

and from the considerations brought forward in the last section, it is clear that the solution is

$$P''_0 = \tfrac{1}{2}\left(\Sigma\frac{x^2}{a^2}\right)^2, \qquad (66)$$

so that

$$L'' = \tfrac{1}{2}a^4, \text{ \&c.}; \; p'' = q'' = r'' = 0.$$

We are left with the problem of determining P′, which must satisfy

$$\int_0^\infty [P'-\tfrac{1}{4}f DP'+\tfrac{1}{64}f^2D^2P']\frac{d\lambda}{\Delta}-\theta P'_0$$
$$= \tfrac{1}{2}(\Sigma J_{AA}x^4+2\Sigma J_{BC}y^2z^2-J)+\Delta n(x^2+y^2). \qquad (67)$$

Unfortunately there is no simple means of dealing with this equation, and the general solution obtained by direct algebraic treatment is so complicated as to convey no meaning at all to the mind. Our method will be to consider first an approximate solution of a simple form, this having reference only to the fourth degree terms in P′; we shall then attempt to estimate the amount of error involved, giving detailed calculations and precise solutions for two configurations of special importance.

16. The approximate solution we shall consider is

$$L' = -\tfrac{1}{2}\frac{a^8}{\theta}J_{AA},\; l' = -\tfrac{1}{2}\frac{b^4c^4}{\theta}J_{BC}, \text{ \&c.} \qquad (68)$$

This satisfies equation (67) as regards terms of fourth degree except for the integral on the left-hand, so that the error of the solution is roughly measured by the value of this integral.

Now from the definition of the integrals J_{AA}, &c., it is clear that the greater part of the value of these integrals arises from contributions from comparatively small values of λ, so that to a moderate approximation we shall have

$$\frac{J_{AA}}{\frac{1}{a^4}} = \frac{J_{BB}}{\frac{1}{b^4}} = \frac{J_{CC}}{\frac{1}{c^4}} = \frac{J_{BC}}{\frac{1}{b^2c^2}} = \ldots = k, \text{ say.} \qquad (69)$$

If these relations were strictly true, the value of P_0' would be

$$P_0' = -\tfrac{1}{2}\frac{k}{\theta}\left(\Sigma\frac{x^2}{a^2}\right)^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

so that, from the consideration of § 14, we should have

$$\int_0^\infty [P' - \tfrac{1}{4}f DP' + \tfrac{1}{64}f^2 D^2P'] \frac{d\lambda}{\Delta} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

Thus the error is of the order of the error of the approximation (69) multiplied by the coefficients of the integral (71), which coefficients are found to be small. It may be noticed that equations (69) are strictly true in the spherical configuration in which $a = b = c$, and the error increases as a, b, c become more unequal.

Thus the error in solution (68) is *nil* in the spherical configuration; we shall now evaluate it exactly in two other configurations: (i.) the point of bifurcation of the spheroidal and ellipsoidal series, and (ii.) the point of bifurcation of the ellipsoidal and pear-shaped series.

17. *Ellipsoidal Point of Bifurcation.*—At the point of bifurcation of the series of spheroids with the series of ellipsoids, the semi-axes a, b, c, and the values of n and θ are given by

$$a = b = 1{\cdot}1972, \qquad c = 0{\cdot}69766,$$

$$n = \frac{\omega^2}{2\pi\rho_0} = 0{\cdot}18712, \qquad \theta = 0{\cdot}47126,$$

the scale of length, which is entirely at our disposal, being chosen so as to make $abc = 1$.

Since the configuration under discussion is now spheroidal, we have

$$L = M = n, \qquad l = m, \qquad p = q,$$

so that there are only three coefficients, say, L, m and N, to the terms of fourth degree, and two coefficients p, r to terms of second degree.

For any configuration, formula (48) expresses the internal potential of a solid of uniform unit density, so that

$$\nabla^2 [V_i(1) + \epsilon \Delta V_i(1) + \epsilon^2 \delta V_i(1) + \ldots] = -4\pi. \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

It follows that $\Delta V_i(1)$, $\delta V_i(1)$, &c., are all spherical harmonics.

In the present problem in which $a = b$, these harmonics are also symmetrical about the axis of z, and so are functions of x^2+y^2 and of z only, so that we may assume

$$\begin{aligned}\int_0^\infty [P - \tfrac{1}{4}f DP + \tfrac{1}{64}f^2 D^2P] \frac{d\lambda}{\Delta}, \\ = -\frac{\Delta V_i(1)}{\pi abc}, \\ = 4c_{11}[(x^2+y^2)^2 - 8(x^2+y^2)z^2 + \tfrac{8}{3}z^4] + 4d_1(x^2+y^2-2z^2), \quad . \quad . \quad (73)\end{aligned}$$

in which the quantities in square brackets are the most general zonal harmonics of appropriate type, and the coefficients c_{11}, d_1, are chosen so as to be identical with those used in my previous paper. Either by direct calculation or by comparison with the results obtained in this previous paper, we find

$$4c_{11} = \mathrm{L}\left(\mathrm{J}_{\mathrm{A}^4}-\frac{4}{a^2}\mathrm{I}_{\mathrm{A}^4}+\frac{1}{a^4}\mathrm{H}_{\mathrm{A}^4}\right)$$
$$+m\left(-\frac{1}{c^2}\mathrm{I}_{\mathrm{A}^3\mathrm{C}}+\frac{1}{2a^2c^2}\mathrm{H}_{\mathrm{A}^3\mathrm{C}}\right)+\frac{3}{8c^4}\mathrm{N}\mathrm{H}_{\mathrm{A}^2\mathrm{C}^2}. \quad . \quad . \quad . \quad . \quad (74)$$

$$4d_1 = 2p\left(\mathrm{J}_{\mathrm{AA}}-\frac{1}{a^2}\mathrm{I}_{\mathrm{AA}}\right)-\frac{r}{c^2}\mathrm{I}_{\mathrm{AC}}$$
$$+\frac{6\mathrm{L}}{a^2}\mathrm{I}_{\mathrm{A}^3}-\frac{2\mathrm{L}}{a^4}\mathrm{I}_{\mathrm{A}^2}+\frac{m}{c^2}\left(2\mathrm{I}_{\mathrm{A}^2\mathrm{C}}-\frac{1}{a^2}\mathrm{I}_{\mathrm{AC}}\right)-\frac{3}{4c^2}\mathrm{N}\left(\frac{1}{c^2}\mathrm{I}_{\mathrm{AC}}-\mathrm{I}_{\mathrm{AC}^2}\right), \quad . \quad . \quad (75)$$

in which the notation is that already defined in formula (37), supplemented by the further abbreviations:

$$\mathrm{I}_{\mathrm{AB}\ldots} = \int_0^\infty \frac{\lambda\, d\lambda}{\Delta\,(a^2+\lambda)\,(b^2+\lambda)\ldots}\,; \qquad \mathrm{H}_{\mathrm{AB}\ldots} = \int \frac{\lambda^2\, d\lambda}{\Delta\,(a^2+\lambda)\,(b^2+\lambda)\ldots}\,, \quad . \quad . \quad (76)$$

and the further abbreviations $\mathrm{G}_{\mathrm{AB}\ldots}$, $\mathrm{F}_{\mathrm{AB}\ldots}$, will be used when required, to denote similar integrals having terms λ^3, λ^4 in the numerator.

18. For computation, it is necessary to construct tables of these integrals. The table on the next page contains values which are required both here and later in the paper. The method of computation has been fully described elsewhere.*

Using these values, I find in place of equation (74)

$$4c_{11} = 0{\cdot}010947\mathrm{L}-0{\cdot}064325m+0{\cdot}094707\mathrm{N}. \quad . \quad . \quad . \quad . \quad . \quad (77)$$

On equating coefficients of x^4, x^2z^2 and z^4 in equation (59) we obtain

$$\left.\begin{aligned} 4c_{11}-\theta\frac{\mathrm{L}}{a^8} &= \tfrac{1}{2}\mathrm{J}_{\mathrm{AA}}-\tfrac{1}{2}(\gamma-2)\frac{\theta}{a^4} \\ -16c_{11}-\theta\frac{m}{a^4c^4} &= \tfrac{1}{2}\mathrm{J}_{\mathrm{AC}}-\tfrac{1}{2}(\gamma-2)\frac{\theta}{a^2c^2} \\ \tfrac{32}{3}c_{11}-\theta\frac{\mathrm{N}}{c^8} &= \tfrac{1}{2}\mathrm{J}_{\mathrm{CC}}-\tfrac{1}{2}(\gamma-2)\frac{\theta}{c^4} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

* 'Phil. Trans.,' A, vol. 215, p. 60.

Table of Integrals.

$(a = 1·19723, \qquad c = 0·69766.)$

	J.	I.	H.	G.	F.
A	0·51589				
C	0·96821				
AA	0·22938	0·18712			
AC	0·47781	0·28334			
CC	1·05115	0·45659			
AAA	0·11850	0·05953	0·10177		
AAC	0·26244	0·10164	0·13766		
ACC	0·60567	0·18302	0·19425		
CCC	1·44620	0·34725	0·28757		
A^4	0·06589	0·02406	0·02506	0·06585	
A^3C	0·15204	0·04451	0·03786	0·08335	
A^2C^2	0·36257	0·08599	0·05983	0·10849	
AC^3	0·88791	0·17349	0·09857	0·14629	
C^4	2·22418	0·36363	0·17026	0·20470	
A^5	0·03828	0·01102	0·00826	0.01322	0·0469
A^4C	0·09100	0·02162	0·01352	0·01848	0·0569
A^3C^2	0·22240	0·04381	0·02321	0·02656	0·0704
A^2C^3	0·55496	0·09243	0·04102	0·03978	0·0893
AC^4	1·41157	0·20087	0·07573	0·06171	0·1162
A^6	0·02290	0·00545	0·00321	0·00366	0·0080
A^5C	0·05570	0·01120	0·00555	0·00557	0·0105
A^4C^2	0·13876	0·02345	0·01025	0·00852	0·0143
A^3C^3	0·35131	0·05136	0·01881	0·01407	0·0198
A^2C^4	0·90490	0·11455	0·03668	0·02315	0·0285
A^7	—	0·00284	0·00138	0·00123	0·0019
A^6C	—	0·00608	0·00247	0·00201	0·0027
A^5C^2	—	0·01294	0·00496	0·00314	0·0040
A^4C^3	—	0·02948	0·00904	0·00586	0·0057
A^2C^4	—	0·06675	0·01887	0·00964	0·0093

Introducing the value (77) for $4c_{11}$, these become three linear equations in L, m and N, of which I find the solution to be

$$\left.\begin{aligned} L = M = n &= 1·0273\,(\gamma-2) - 1·0466 \\ l = m &= 0·3488\,(\gamma-2) - 0·2378 \\ N &= 0·11845\,(\gamma-2) - 0·06328 \end{aligned}\right\}\text{(exact solution).} \qquad (79)$$

This may be compared with the approximate solution obtained in § 16, which

may be referred to as "Approximation A." Inserting numerical values, this stands as follows:—

$$\left.\begin{aligned} \mathrm{L} = \mathrm{M} = n &= 1{\cdot}0273\,(\gamma-2)-1{\cdot}0273 \\ l = m &= 0{\cdot}3488\,(\gamma-2)-0{\cdot}2467 \\ \mathrm{N} &= 0{\cdot}11845\,(\gamma-2)-0{\cdot}0626 \end{aligned}\right\}\text{(Approximation A).} \quad (80)$$

It will be observed that the error in Approximation A is greatest when $\gamma = 2$, when it is of the order of 2 per cent. of the whole value.

19. *Pear-Shaped Point of Bifurcation.*—We proceed now to evaluate the exact solution for the configuration at which the Jacobian ellipsoid gives place to the pear-shaped figure. At this point we have

$$a = 1{\cdot}88583, \qquad b = 0{\cdot}814975, \qquad c = 0{\cdot}650659,$$

$$n = \frac{\omega^2}{2\pi\rho_0} = 0{\cdot}141999, \qquad \theta = 0{\cdot}413607,$$

the lengths again being chosen so that $abc = 1$. The values of the integrals necessary for the evaluation of the potentials have been given in previous papers.* The values of L′, M′, N′, l', m', n' are now distinct, and are determined by six equations of the type (*cf.* equations (78))

$$\left.\begin{aligned} 4c_{11}-\theta\left(\frac{\mathrm{L}'}{a^8}\right) &= \tfrac{1}{2}\mathrm{J}_{\mathrm{AA}}, \text{ \&c.} \\ 4c_{23}-\theta\left(\frac{2l'}{b^4c^4}\right) &= \mathrm{J}_{\mathrm{BC}}, \text{ \&c.} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

in which the coefficients c_{11}, c_{22}, c_{33}, c_{12}, c_{23} and c_{31} are now distinct, the condition that the potential shall be harmonic being expressed by three linear relations connecting them. The values of these six potential coefficients have been calculated in my previous paper; inserting these and solving the six equations (81) I find the set of values given in the second column of the following table, the values given by "Approximation A" being given in the adjacent column for comparison:—

Coefficient.	Exact solution.	Approximation A.	Difference per cent.
L′	− 15·4353	− 10·1768	− 34
M′	− 0·15560	− 0·15329	− 1
N′	− 0·04733	− 0·04677	− 1
l'	− 0·08468	− 0·08424	0
m'	− 0·49850	− 0·62860	+ 26
n'	− 0·95103	− 1·18103	+ 24

* 'Phil. Trans.,' A, vol. 215, p. 61, and vol. 217, p. 21.

Clearly the error is quite large, but is concentrated in the coefficients L', m', n' which multiply terms in x. It is remarkable that the cross section of the figure by the plane $x = 0$ is given accurately to within 1 per cent. by this approximation.

20. It appears that Approximation A will give a solution accurate to within 2 per cent. for spheroidal figures, but of error varying from 2 to about 20 per cent. for ellipsoidal figures. A tolerable approximation to any required ellipsoidal figure could perhaps be got by regarding Approximation A as a first approximation, and obtaining the error in this by interpolation between the two errors which have been accurately estimated.

21. A second approximate solution, which has an interest other than that of accuracy, may be referred to here. We may use the approximate equations (69) to simplify the approximate solution (68), and so obtain a still less accurate approximation, which we shall call Approximation B. The approximation is

$$L' = -\frac{k}{2\theta}a^4, \qquad l' = -\frac{k}{2\theta}b^2c^2, \text{ \&c.,} \quad \ldots \ldots \quad (82)$$

and the complete value of P_0 becomes

$$P_0 = \tfrac{1}{2}\left[(\gamma-2)-\frac{k}{\theta}\right]\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^2. \quad \ldots \ldots \quad (83)$$

In this approximation the quantity k is at our choice; we must select it so as to give as good an average value as possible for the approximately equal quantities which occur in equations (69).

Choosing a suitable value of k, I find for the coefficients at the ellipsoidal point of bifurcation

$$\left.\begin{aligned} L = M = n &= 1{\cdot}0273\,(\gamma-2)-0{\cdot}7705 \\ l = m &= 0{\cdot}3488\,(\gamma-2)-0{\cdot}2616 \\ N &= 0{\cdot}11845\,(\gamma-2)-0{\cdot}0888 \end{aligned}\right\}\text{(Approximation B).} \quad \ldots \quad (84)$$

Comparing this with the exact solution given in equations (79), it appears that the error is as great as 20 per cent. when $\gamma = 2$.

The corresponding approximation at the pear-shaped point of bifurcation is almost worthless, the error being one of fully 50 per cent.

The significance of these approximations will appear later.

22. We turn now to the evaluation of the terms of second degree in x, y, z. It has already been seen (p. 172) that $p'' = q'' = r'' = 0$, so that p, q, r are identical with p', q', r'. Or, in other words, p, q, r do not involve $(\gamma-2)$.

Let us suppose that in the general ellipsoidal solution (*cf.* equation (73))

$$\int_0^\infty [P-\tfrac{1}{4}f\,DP+\tfrac{1}{64}f^2\,D^2P]\frac{d\lambda}{\Delta} = \text{fourth degree terms}+4\,(d_1x^2+d_2y^2+d_3z^2). \quad . \quad (85)$$

Then on equating coefficients of x^2, y^2 and z^2 in equation (59) we obtain

$$4d_1 - \frac{2p\theta}{a^4} = \Delta n, \quad \ldots \ldots \quad (86)$$

$$4d_2 - \frac{2q\theta}{b^4} = \Delta n, \quad \ldots \ldots \quad (87)$$

$$4d_3 - \frac{2r\theta}{c^4} = 0. \quad \ldots \ldots \quad (88)$$

The value of $4d_1$ is readily found to be*

$$4d_1 = 2p\mathrm{J}_{\mathrm{AA}} - \frac{p}{a^2}\mathrm{I}_{\mathrm{AA}} - \frac{q}{b^2}\mathrm{I}_{\mathrm{AB}} - \frac{r}{c^2}\mathrm{I}_{\mathrm{AC}} + 4e_1, \quad \ldots \ldots \quad (89)$$

where

$$4e_1 = -\tfrac{3}{4}\left(\frac{\mathrm{L}}{a^4}\mathrm{H}_{\mathrm{A}^3} + \frac{\mathrm{M}}{b^4}\mathrm{H}_{\mathrm{AB}^2} + \frac{\mathrm{N}}{c^4}\mathrm{H}_{\mathrm{AC}^2}\right) - \tfrac{1}{2}\left(\frac{l}{b^2c^2}\mathrm{H}_{\mathrm{ABC}} + \frac{m}{c^2a^2}\mathrm{H}_{\mathrm{AAC}} + \frac{n}{a^2b^2}\mathrm{H}_{\mathrm{AAB}}\right)$$
$$+ \frac{3\mathrm{L}}{a^2}\mathrm{I}_{\mathrm{A}^3} + \frac{n}{b^2}\mathrm{I}_{\mathrm{A}^2\mathrm{B}} + \frac{m}{c^2}\mathrm{I}_{\mathrm{A}^2\mathrm{C}}. \quad \ldots \ldots \quad (90)$$

The values of $4d_2$ and $4d_3$ can of course be written down from symmetry.

It is immaterial whether we insert the values of L, M, N, ..., or of L′, M′, N′, ..., in these equations in evaluating $4e_1$, &c., since the contributions from L″, M″, N″, ..., which are proportional to a^4, b^4, c^4, ..., must vanish by § 14.

23. If we are content to use Approximation B, the solution of our problem assumes a very simple form. For L′, M′, N′, ... are now proportional to a^4, b^4, c^4, ..., and therefore

$$e_1 = e_2 = e_3 = 0 \text{ (Approximation B)}. \quad \ldots \ldots \quad (91)$$

Thus equations (86), &c., reduce to

$$2p\mathrm{J}_{\mathrm{AA}} - \frac{p}{a^2}\mathrm{I}_{\mathrm{AA}} - \frac{q}{b^2}\mathrm{I}_{\mathrm{AB}} - \frac{r}{c^2}\mathrm{I}_{\mathrm{AC}} - \frac{2p\theta}{a^4} = \Delta n, \quad \ldots \ldots \quad (92)$$

and two similar equations.

The quantity Δn, which has been introduced for convenience only, is entirely at our disposal. Let us take $\Delta n = 0$, then the solution of the equations is

$$p = q = r = 0 \text{ (Approximation B)}. \quad \ldots \ldots \quad (93)$$

Thus, corresponding to Approximation B, the solution corresponding to rotation ω given by $\omega^2/2\pi\rho_0 = n$ has for density

$$\rho = \rho_0 - (\rho_0 - \sigma)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) - \frac{(\rho_0 - \sigma)^2}{2\rho}\left(\frac{k}{\theta} - (\gamma - 2)\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)^2, \quad . \quad (94)$$

* *Cf.* 'Phil. Trans.,' A, vol. 215, p. 60.

while the boundary is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1-\frac{\rho_0-\sigma}{2\rho_0}\left(\frac{k}{\theta}-(\gamma-2)\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

and so is still ellipsoidal. So far as this approximation goes, it appears that the shape of the ellipsoid, as given by the ratio of the semi-axes, depends only on $\omega^2/2\pi\rho_0$; it is the same as for a homogeneous mass of uniform density ρ_0, equal to the density at the centre of the compressible mass, rotating with the same angular velocity.

24. When this somewhat unsatisfactory approximation is abandoned there is no means of procedure except to calculate e_1, e_2 and e_3 directly from equations (89), &c.

Suppose that instead of being given by equation (33) the tidal potential had been given by

$$\mathrm{V_T}+\epsilon\,\Delta\mathrm{V_T} = (\tau_1+\epsilon\Delta\tau_1)\,x^2+(\tau_2+\epsilon\Delta\tau_2)\,y^2+(\tau_3+\epsilon\Delta\tau_3)\,z^2. \quad . \quad . \quad . \quad (96)$$

Then on equating terms independent of ϵ in the principal equation we should obtain the same equation as before, but in place of equations (86) to (88), there would be three equations such as

$$4d_1-\frac{2p\theta}{a^4} = \Delta n+\frac{\Delta\tau_1}{\pi\rho_0 abc},$$

or, inserting the value of $4d_1$ from equation (89),

$$2p\mathrm{J_{AA}}-\frac{p}{a^2}\mathrm{I_{AA}}-\frac{q}{b^2}\mathrm{I_{AB}}-\frac{r}{c^2}\mathrm{I_{AC}}+4e_1-\frac{2p\theta}{a^4} = \Delta n+\frac{\Delta\tau_1}{\pi\rho_0 abc}. \quad . \quad . \quad . \quad (97)$$

These equations become identical with equations (92) if we take

$$\Delta\tau_1 = 4\pi\rho_0 abce_1,\ \text{\&c.}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

and the equations then have the simple solution

$$p = q = r = 0, \qquad \Delta n = 0.$$

Thus the exact solution can be found by superposing the fourth degree terms already calculated on to an ellipsoid which is a figure of equilibrium under an additional tidal potential

$$\epsilon\,\Delta\mathrm{V_T} = 4\pi\rho_0 abc\epsilon\,(e_1x^2+e_2y^2+e_3z^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (99)$$

this necessarily being harmonic, since $e_1+e_2+e_3 = 0$.

25. Let a', b', c' be the semi-axes of a figure of equilibrium under this additional tidal force, these differing from the old quantities a, b, c by small quantities of the order of ϵ. The equations determining a', b', c' are (*cf.* equations (38) to (41))

$$\mathrm{J'_A}-\frac{\tau_1}{\pi\rho_0 a'b'c'}+4\epsilon e_1-\frac{\omega'^2}{2\pi\rho_0 a'b'c'} = \frac{\theta'}{a'^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (100)$$

and two similar equations, accented letters all referring to the new figure of equilibrium, and the value of θ' being now given by (*cf.* equation (41))

$$\theta' = \frac{\kappa\gamma\epsilon\rho_0^{\gamma-2}}{\pi a'b'c'}, \quad \ldots \ldots \ldots \quad (101)$$

so that, by comparison with equation (41),

$$\theta' a'b'c' = \theta abc. \quad \ldots \ldots \ldots \quad (102)$$

There will, of course, be a solution for the compressible mass corresponding to each solution of these equations, which are virtually equations giving a solution of an incompressible mass under tidal forces.

When the main tidal force disappears ($\tau_1 = \tau_2 = \tau_3 = 0$) the solutions will correspond, except for small tidal terms, to Maclaurin spheroids and Jacobian ellipsoids. The compressible solution which corresponds to the Maclaurin-Jacobian point of bifurcation will represent a point of bifurcation for the compressible mass. Let us proceed to determine this point of bifurcation exactly.

At the point of bifurcation the configuration is spheroidal, so that $e_1 = e_2$. By subtraction of corresponding sides of equation (100) and the similar equation in b, we obtain

$$J'_A - J'_B = \theta'\left(\frac{1}{a'^2} - \frac{1}{b'^2}\right),$$

or

$$(a'^2 - b'^2)(J'_{AB} - \theta'/a'^2b'^2) = 0. \quad \ldots \ldots \ldots \quad (103)$$

The spheroidal series is determined by the vanishing of the first factor, and the ellipsoidal series by the vanishing of the second factor. At the point of bifurcation both factors vanish, so that $a' = b'$, and

$$a'^4 J'_{AA} = \theta',$$

or again, using relation (102),

$$a'^6 c' J'_{AA} = \theta a^2 c. \quad \ldots \ldots \ldots \quad (104)$$

This equation, together with (100) and its companion, will determine the values of a', c' and ω' at the point of bifurcation.

26. The values of these quantities differ by terms of the order of ϵ from the corresponding quantities a, b, c at the Maclaurin-Jacobian point of bifurcation, so that we may suppose that

$$\frac{1}{a'^2} = \frac{1}{a^2} + \epsilon\,\Delta\left(\frac{1}{a^2}\right), \text{ \&c.,} \qquad \omega'^2 = \omega^2 + \epsilon\,\Delta\omega^2. \quad \ldots \quad (105)$$

The equation of the figure of equilibrium is now

$$\frac{x^2+y^2}{a^2} + \frac{y^2}{b^2} + \epsilon\left[(x^2+y^2)\,\Delta\left(\frac{1}{a^2}\right) + z^2\,\Delta\left(\frac{1}{c^2}\right) + \frac{Lx^4}{a^8} + \ldots\right],$$

and this agrees exactly with the figure previously considered if we take

$$\Delta\left(\frac{1}{a^2}\right) = \frac{2p}{a^4}; \qquad \Delta\left(\frac{1}{c^2}\right) = \frac{2r}{c^4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

Using relations (105) and (106), equation (100) transforms to

$$abc\left[\mathrm{J}_{\mathrm{A}} - \frac{\tau_1}{\pi\rho_0 abc} - \frac{\omega^2}{2\pi\rho_0 abc} - \frac{\theta}{a^2}\right]$$

$$+\epsilon abc\left[4e_1 + p\left(2\mathrm{J}_{\mathrm{AA}} - \frac{2}{a^2}\mathrm{I}_{\mathrm{AA}}\right) - \frac{r}{c^2}\mathrm{I}_{\mathrm{AC}} - \frac{\Delta\omega^2}{2\pi\rho_0 abc} - \frac{2p\theta}{a^4}\right] = 0. \quad . \quad . \quad (107)$$

The first line vanishes in virtue of equation (38) and the equation reduces to

$$4e_1 + p\left(2\mathrm{J}_{\mathrm{AA}} - \frac{2}{a^2}\mathrm{I}_{\mathrm{AA}}\right) - \frac{r}{c^2}\mathrm{I}_{\mathrm{AC}} - \frac{2p\theta}{a^4} = \Delta n, \quad . \quad . \quad . \quad . \quad . \quad (108)$$

which agrees with equation (86), as of course it ought to do, and there is a similar equation agreeing with equation (88).

27. Equation (104) may be written in the slightly symbolical form

$$a^6 c\mathrm{J}_{\mathrm{AA}} + \epsilon\,\Delta\left(a^6 c\mathrm{J}_{\mathrm{AA}}\right) = \theta a^2 c, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

and the condition which determines the ordinary Maclaurin-Jacobian point of bifurcation is

$$a^4\mathrm{J}_{\mathrm{AA}} = \theta.$$

Thus equation (109), which expresses the condition to be satisfied at the point of bifurcation on the compressible series, reduces to

$$\Delta\left(a^6 c\mathrm{J}_{\mathrm{AA}}\right) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (110)$$

Now from the definition of J_{AA} we have

$$a^6 c\mathrm{J}_{\mathrm{AA}} = \int_0^\infty \frac{d\lambda}{(1+\lambda/a^2)^3(1+\lambda/c^2)^{\frac{1}{2}}},$$

whence we readily find that equation (110) is equivalent to

$$6\mathrm{I}_{\mathrm{A}^3}\left(\frac{p}{a^2}\right) + \mathrm{I}_{\mathrm{A}^2\mathrm{C}}\left(\frac{r}{c^2}\right) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (111)$$

Equation (111), together with equation (108) and its companion, determine the values of p, r and Δn at the point of bifurcation on the series of compressible configurations.

28. The material for numerical computations has already been given in § 18. Using the exact solution there given, I find, by direct computation from equation (90),

$$4e_1 = 4e_2 = -0{\cdot}03404\,; \qquad 4e_3 = 0{\cdot}06808. \quad \ldots \ldots \quad (112)$$

From equation (111), and the companion to (108), I now find

$$\frac{p}{a^2} = -0{\cdot}016037\,; \qquad \frac{r}{c^2} = 0{\cdot}056337, \quad \ldots \ldots \quad (113)$$

and equation (108) now gives

$$\Delta n = \frac{\Delta\omega^2}{2\pi\rho_0 abc} = -0{\cdot}04400. \quad \ldots \ldots \quad (114)$$

These quantities are all comparatively small; they would all have vanished under Approximation B.

Second Order Solution.

29. We now pass to the solution of higher order still. On equating terms in ϵ^2 in equation (50), we obtain

$$\frac{1}{\pi abc}\left(\delta V_i(1) - \Delta E_i\right) + \delta n\,(x^2+y^2)$$

$$= -\theta\left[Q_0 - (\gamma-2)\,P_0\left(\Sigma\frac{x^2}{a^2}\right) + \tfrac{1}{6}(\gamma-2)(\gamma-3)\left(\Sigma\frac{x^2}{a^2}\right)^3\right]. \quad . \quad (115)$$

30. From equation (28), taking terms as far as first powers of ϵ only,

$$E_i + \Delta E_i = \pi abc\int_0^\infty\left[\int_q^1 q^2\left(1 - \frac{\partial\phi(q)}{\partial q^2}\right)dq^2\right]\frac{d\lambda}{\Delta}, \quad \ldots \ldots \quad (116)$$

in which the lower limit q is given by

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} + \phi(q) = q^2 \quad \ldots \ldots \quad (117)$$

As far as ϵ, equation (12) gives

$$\phi(q) = \epsilon\left[P - \tfrac{1}{4}fDP + \tfrac{1}{64}f^2D^2P\right], \quad \ldots \ldots \quad (118)$$

while from § 8,

$$q^2 f = \frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} - q^2 \quad \ldots \ldots \quad (119)$$

$$\frac{D}{q^2} = \left(\frac{1}{a^2} - \frac{1}{a^2+\lambda}\right)\frac{\partial^2}{\partial\xi^2} + \left(\frac{1}{b^2} - \frac{1}{b^2+\lambda}\right)\frac{\partial^2}{\partial\eta^2} + \left(\frac{1}{c^2} - \frac{1}{c^2+\lambda}\right)\frac{\partial^2}{\partial\zeta^2}. \quad \ldots \quad (120)$$

The value of D/q^2 is independent of q, and $\phi(q)$ may be rearranged in the form

$$\phi(q) = \epsilon\left[P - \tfrac{1}{4}(q^2 f)\frac{D}{q^2}P + \tfrac{1}{64}(q^2 f)^2\left(\frac{D}{q^2}\right)^2 P\right],$$

whence, on differentiation

$$\frac{\partial\phi(q)}{\partial q^2} = \epsilon\left[\tfrac{1}{4}\frac{D}{q^2}P - \tfrac{1}{32}(q^2 f)\left(\frac{D}{q^2}\right)^2 P\right],$$

so that

$$q^2\left(1 - \frac{\partial\phi(q)}{\partial q^2}\right) = q^2\left[1 - \frac{\epsilon}{4}\frac{D}{q^2}P + \frac{\epsilon}{32}\left(\Sigma\frac{x^2}{a^2+\lambda}\right)\left(\frac{D}{q^2}\right)^2 P\right] - \frac{\epsilon}{32}q^4\left(\frac{D}{q^2}\right)^2 P.$$

This gives on integration, since D/q^2 is independent of q,

$$\int_q^1 q^2\left(1 - \frac{\partial\phi(q)}{\partial q^2}\right)dq^2 = \tfrac{1}{2}(1-q^4)\left[1 - \frac{\epsilon}{4}\frac{D}{q^2}P + \frac{\epsilon}{32}\left(\Sigma\frac{x^2}{a^2+\lambda}\right)\left(\frac{D}{q^2}\right)^2 P\right]$$
$$- \frac{\epsilon}{96}(1-q^6)\left(\frac{D}{q^2}\right)^2 P. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (121)$$

The value of q^2 on the right of this equation is the root of equation (117), $\phi(q)$ being given by equation (118) in which f is in turn given by equation (119). Thus q^2 will be obtained by the elimination of $\phi(q)$ and f from equations (117) to (119), omitting terms in ϵ^2. From these three equations we obtain

$$q^2 f = -\phi(q) = -\epsilon\left[P - \tfrac{1}{4}fDP + \tfrac{1}{64}f^2D^2P\right],$$

so that f is a small quantity of the order of ϵ, and, omitting terms in ϵ^2,

$$\phi(q) \doteq \epsilon P,$$

which is independent of q. Equation (117) now gives, as the value of q^2,

$$q^2 = \Sigma\frac{x^2}{a^2+\lambda} + \epsilon P. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (122)$$

The coefficient of ϵ on the right hand of equation (121) now reduces to

$$-P\Sigma\frac{x^2}{a^2+\lambda} - \left[1 - \left(\Sigma\frac{x^2}{a^2+\lambda}\right)^2\right]\left[\tfrac{1}{8}\frac{D}{q^2}P - \tfrac{1}{64}\left(\Sigma\frac{x^2}{a^2+\lambda}\right)\left(\frac{D}{q^2}\right)^2 P\right]$$
$$- \tfrac{1}{96}\left[1 - \left(\Sigma\frac{x^2}{a^2+\lambda}\right)^3\right]\left(\frac{D}{q^2}\right)^2 P.$$

The value of P is given by equation (56), and that of D/q^2 by equation (120). From these we obtain

$$\frac{D}{q^2}P = \Sigma\left(\frac{1}{a^2} - \frac{1}{a^2+\lambda}\right)(12L\xi^2 - 4n\eta^2 + 4m\zeta^2 + 4p),$$

$$\left(\frac{D}{q^2}\right)^2 P = \Sigma\left[24L\left(\frac{1}{a^2} - \frac{1}{a^2+\lambda}\right)^2 + 16l\left(\frac{1}{b^2} - \frac{1}{b^2+\lambda}\right)\left(\frac{1}{c^2} - \frac{1}{c^2+\lambda}\right)\right],$$

whence we readily obtain, writing A for $a^2+\lambda$ and so on,

$$\Delta E_i = -\pi abc\int_0^\infty\left\{\left(\Sigma\frac{x^2}{A}\right)\left(\Sigma L\frac{x^4}{A^4}+2\Sigma l\frac{y^2z^2}{B^2C^2}+2\Sigma p\frac{x^2}{A^2}\right)\right.$$

$$+\tfrac{1}{8}\left[1-\left(\Sigma\frac{x^2}{A}\right)^2\right]\left[\Sigma\frac{\lambda}{a^2A}\left(12L\frac{x^2}{A^2}+4n\frac{y^2}{B^2}+4m\frac{z^2}{C^2}+4p\right)\right]$$

$$\left.+\left[\tfrac{1}{96}-\tfrac{1}{64}\left(\Sigma\frac{x^2}{A}\right)+\tfrac{1}{192}\left(\Sigma\frac{x^2}{A}\right)^3\right]\left[24\Sigma\frac{L\lambda^2}{a^4A^2}+16\Sigma\frac{l\lambda^2}{b^2c^2BC}\right]\right\}\frac{d\lambda}{\Delta}. \quad (123)$$

31. From § 8, the value of $V_i(1)$ is given by

$$V_i(1)+\epsilon\Delta V_i(1)+\epsilon^2\partial V_i(1) = -\pi abc\int_0^\infty[f+\phi(1)]\frac{d\lambda}{\Delta}, \quad . \quad . \quad . \quad (124)$$

in which the value of $\phi(1)$ is now (*cf.* equations (12) and (47))

$$\phi(1) = \epsilon[P-\tfrac{1}{4}fDP+\tfrac{1}{64}f^2D^2P]$$
$$-\tfrac{1}{8}\epsilon^2[DP^2-\tfrac{1}{8}fD^2P^2+\tfrac{1}{192}f^2D^3P-\tfrac{1}{9216}f^3D^4P^2]$$
$$+\epsilon^2[Q-\tfrac{1}{4}fDQ+\tfrac{1}{64}f^2D^2Q-\tfrac{1}{2304}f^3D^3Q+\ldots]$$
$$+\text{terms in }\epsilon^3,\ \&c. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (125)$$

Thus the value of $\delta V_i(1)$, being the coefficient of ϵ^2 on the right hand of equation (124), is given by

$$\delta V_i(1) = -\pi abc\int_0^\infty[Q-\tfrac{1}{4}fDQ+\tfrac{1}{64}f^2D^2Q-\tfrac{1}{2304}f^3D^3Q+\ldots]\frac{d\lambda}{\Delta}$$
$$+\tfrac{1}{8}\pi abc\int_0^\infty[DP^2-\tfrac{1}{8}fD^2P^2+\tfrac{1}{192}f^2D^3P^2-\tfrac{1}{9216}f^3D^4P^2]\frac{d\lambda}{\Delta}.$$

32. Collecting and rearranging terms equation (115) now becomes

$$\int_0^\infty[Q-\tfrac{1}{4}fDQ+\tfrac{1}{64}f^2D^2Q-\tfrac{1}{2304}f^3D^3Q+\ldots]\frac{d\lambda}{\Delta}-\theta Q_0$$
$$=\theta(\gamma-2)\left[\tfrac{1}{6}(\gamma-3)\left(\Sigma\frac{x^2}{a^2}\right)^3-[P'_0+(\gamma-2)P''_0]\left(\Sigma\frac{x^2}{a^2}\right)\right]$$
$$+\tfrac{1}{8}\int_0^\infty[DP^2-\tfrac{1}{8}fD^2P^2+\tfrac{1}{192}f^2D^3P^2-\tfrac{1}{9216}f^3D^4P^2]\frac{d\lambda}{\Delta}$$
$$-\frac{\Delta E_i}{\pi abc}+\delta n(x^2+y^2). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (126)$$

This is an equation for the determination of Q, all the terms involving Q being on the left hand, and all the terms independent of Q on the right. The terms on the right are all of degrees 6, 4, and 2 in x, y, z, so that the value of Q will clearly consist of terms of degrees 6, 4, and 2 in x, y, z. Further, from the linearity of the

equation, it is clear that the value of Q will be the sum of a number of contributions arising from the separate terms on the right.

Consider now what is the contribution from the first term on the right. Or, in other words, consider what would be the solution if the whole equation were reduced to

$$\int_0^\infty [Q-\tfrac{1}{4}f\,DQ+\tfrac{1}{64}f^2\,D^2Q-\tfrac{1}{2304}f^3\,D^3Q+\ldots]\frac{d\lambda}{\Delta}-\theta Q_0$$
$$= \tfrac{1}{6}\theta(\gamma-2)(\gamma-3)\left(\Sigma\frac{x^2}{a^2}\right)^3.$$

The solution would be

$$Q_0 = -\tfrac{1}{6}(\gamma-2)(\gamma-3)\left(\Sigma\frac{x^2}{a^2}\right)^3, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (127)$$

for, in accordance with the principles arrived at in § 14, this value makes the integral on the left vanish, and the equation is then satisfied.

Similarly the contribution from the term in P''_0 on the right is

$$Q_0 = (\gamma-2)^2\,P''_0\left(\Sigma\frac{x^2}{a^2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (128)$$

for P''_0 is equal to $\frac{1}{2}(\Sigma x^2/a^2)^2$, and so the integral on the left again vanishes.

The contribution from the term in P'_0 in equation (126) cannot be so simply evaluated. If, however, we are content to use Approximation B for P'_0, then P'_0 becomes proportional to P''_0, and an approximate solution is

$$Q_0 = (\gamma-2)\,P'_0\left(\Sigma\frac{x^2}{a^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (129)$$

On referring back to § 16, it becomes clear that the accuracy of the approximate solution is of the same order as that which we previously called Approximation A.

Again, if we use Approximation B for P_0, the contribution from the whole second line in equation (126) will be

$$Q_0 = 0,$$

for the whole second line in question now represents merely the second order terms in the potential of a thick homeoidal shell, and so vanishes.

We proceed next to the contribution from the term in ΔE_i. From equation (123) it appears that if we use Approximation B for the values of L, M, N, ..., and also use equations of the type of (69), the first line in ΔE_i, as given by equation (123), becomes a function of $(\Sigma x^2/a^2)$, while the remaining lines vanish.

It follows that an approximate solution is

$$Q_0 = \frac{\Delta E_i}{\pi abc\theta}.$$

There remains the contribution from the term $\delta n\,(x^2+y^2)$ on the right of equation (126). This contribution will consist of second degree terms in Q_0, which represent merely a step along the fundamental series of ellipsoids to allow for the altered rotation, and the requisite terms can be easily calculated.

Collecting all the approximate partial solutions which have been obtained, we find the complete approximate solution

$$Q_0 = -(\gamma-2)\left[\tfrac{1}{6}(\gamma-3)\left(\Sigma\frac{x^2}{a^2}\right)^3 - [P'_0+(\gamma-2)\,P''_0]\left(\Sigma\frac{x^2}{a^2}\right)\right]$$

$$+\frac{\Delta E_i}{\pi abc\theta} + \text{ terms in } \delta n. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (130)$$

Clearly this approximation is of a degree of accuracy comparable with that of our previous Approximation A for P_0, with which it may be compared. This former approximation can be put in the form

$$P_0 = \tfrac{1}{2}(\gamma-2)\left(\Sigma\frac{x^2}{a^2}\right)^2 + \frac{E_i}{\pi abc\theta} + \text{ terms in } \Delta n. \quad . \quad . \quad . \quad . \quad . \quad (131)$$

33. A less good approximation, comparable with the former Approximation B, can be obtained by further simplifying equation (130) by the help of approximate equations such as (69). The value of Q_0 simplified in this way is found to reduce to a function of $(\Sigma x^2/a^2)$, so that the whole solution becomes ellipsoidal.

34. The accurate solution for Q_0 may be supposed to be

$$Q_0 = Q'_0 + Q''_0\,(\gamma-2),$$

where $Q''_0\,(\gamma-2)$ is the part of Q_0 which is accurately given by formulæ (127) and (128); thus

$$Q''_0 = [\tfrac{1}{3}(\gamma-2)+\tfrac{1}{6}]\left(\Sigma\frac{x^2}{a^2}\right)^3.$$

I have calculated the value of Q'_0 accurately for one configuration only, namely, the ellipsoidal point of bifurcation. At this point $a = b$ and x, y enter only through x^2+y^2. Let us write

$$x^2+y^2 = w^2\,; \quad \xi^2+\zeta^2 = \varpi^2.$$

The general value of Q_0 will be of the form

$$Q_0 = \frac{R}{a^{12}}w^6 + \frac{3S}{a^8c^4}w^4z^2 + \frac{3T}{a^4c^8}w^2z^4 + \frac{U}{c^{12}}z^6 + \frac{r}{a^8}w^4 + \frac{2s}{a^4c^4}w^2z^2 + \frac{t}{c^8}z^4 + \frac{2u}{a^4}w^2 + \frac{2v}{c^4}z^2.$$

We may further put

$$R = R' + R''\,(\gamma-2), \text{ \&c.,}$$

then the values of R'', &c., are

$$R'' = [\tfrac{1}{3}(\gamma-2)+\tfrac{1}{6}]\,a^6, \text{ \&c.,}$$

and the value of Q'_0 is

$$Q'_0 = \frac{R'}{a^{12}} w^6 + \frac{3S'}{a^8 c^4} w^4 z^2 + \ldots,$$

so that

$$Q' = R'\varpi^6 + 3S'\varpi^4\zeta^2 + 3T'\varpi^2\zeta^4 + U'\zeta^6 + r'\varpi^4 + 2s'\varpi^2\zeta^2 + t'\zeta^4 + 2u'\varpi^2 + 2v'\zeta^2. \quad . \quad (132)$$

Let us now assume that

$$\int_0^\infty (Q' - \tfrac{1}{4} f\, DQ' + \tfrac{1}{64} f^2\, D^2Q' - \tfrac{1}{2304} f^3\, D^3Q') \frac{d\lambda}{\Delta}$$
$$= h_1(5w^6 - 90w^4z^2 + 120w^2z^2 - 16z^4) + h_2(3w^4 - 24w^2z^2 + 8z^4) + h_3(w^2 - 2z^2), \quad . \quad (133)$$

this being the most general form possible, since the integral in question is necessarily a zonal harmonic. In calculating the value of DQ', we may use the transformation

$$D = \frac{\lambda}{a^2 A}\frac{1}{\varpi}\frac{\partial}{\partial\varpi}\left(\varpi\frac{\partial}{\partial\varpi}\right) + \frac{\lambda}{c^2 C}\frac{\partial^2}{\partial\zeta^2},$$

and on equating coefficients in equation (133), we obtain

$$5h_1 = \int_0^\infty \left[\frac{R'}{A^6} - \frac{1}{4A^5}\left(\frac{\lambda}{a^2A}36R' + \frac{\lambda}{c^2C}6S'\right) + \frac{1}{64A^4}\left(\frac{\lambda^2}{a^4A^2}576R' + \frac{\lambda^2}{a^2c^2AC}192S' + \frac{\lambda^2}{c^4C^2}72T'\right)\right.$$
$$\left. - \frac{1}{2304A^3}\left(\frac{\lambda^3}{a^6A^3}2304R' + \frac{3\lambda^3}{a^4c^2A^2C}384S' + \frac{3\lambda^3}{a^2c^4AC^2}288T' + \frac{\lambda^3}{c^6C^3}720U'\right)\right]\frac{d\lambda}{\Delta},$$
$$. \quad . \quad . \quad . \quad (134)$$

$$3h_2 = \int_0^\infty \left[\frac{r'}{A^4} - \frac{1}{4A^3}\left(\frac{\lambda}{a^2A}16r' + \frac{\lambda}{c^2C}4s'\right) + \frac{1}{4A^4}\left(\frac{\lambda}{a^2A}36R' + \frac{\lambda}{c^2C}6S'\right)\right.$$
$$+ \frac{1}{64A^2}\left(\frac{\lambda^2}{a^4A^2}64r' + \frac{2\lambda^2}{a^2c^2AC}16s' + \frac{\lambda^2}{c^4C^2}24t'\right)$$
$$- \frac{1}{32A^3}\left(\frac{\lambda^2}{a^4A^2}576R' + \frac{2\lambda^2}{a^2c^2AC}96S' + \frac{\lambda^2}{c^4C^2}72T'\right)$$
$$\left. + \frac{3}{2304A^2}\left(\frac{\lambda^3}{a^6A^3}2304R' + \frac{3\lambda^3}{a^4c^2A^2C}384S' + \frac{3\lambda^3}{a^2c^4AC^2}288T' + \frac{\lambda^3}{c^6C^3}720U'\right)\right]\frac{d\lambda}{\Delta},$$
$$. \quad . \quad . \quad . \quad (135)$$

$$h_3 = \int_0^\infty \left[\frac{2u'}{A^2} - \frac{1}{4A}\left(\frac{\lambda}{a^2A}8u' + \frac{\lambda}{c^2C}4v'\right) + \frac{1}{4A^2}\left(\frac{\lambda}{a^2A}16r' + \frac{\lambda}{c^2C}4s'\right)\right.$$
$$- \frac{1}{32A}\left(\frac{\lambda^2}{a^4A^2}64r' + \frac{2\lambda^2}{a^2c^2AC}16s' + \frac{\lambda^2}{c^4C^2}24t'\right)$$
$$+ \frac{1}{64A^2}\left(\frac{\lambda^2}{a^4A^2}576R' + \frac{2\lambda^2}{a^2c^2AC}96S' + \frac{\lambda^2}{c^4C^2}72T'\right)$$
$$\left. - \frac{3}{2304A}\left(\frac{\lambda^3}{a^6A^3}2304R' + \frac{3\lambda^3}{a^4c^2A^2C}384S' + \frac{3\lambda^3}{a^2c^4AC^2}288T' + \frac{\lambda^3}{c^6C^3}720U'\right)\right]\frac{d\lambda}{\Delta},$$
$$. \quad . \quad . \quad . \quad (136)$$

35. The value of P is

$$P = L\varpi^4 + 2m\varpi^2\zeta^2 + N\zeta^4 + 2p\varpi^2 + 2r\zeta^2,$$

whence we find

$$DP^2 = \mathfrak{R}\varpi^6 + 3\mathfrak{S}\varpi^4\zeta^2 + 3\mathfrak{T}\varpi^2\zeta^4 + \mathfrak{U}\zeta^6 + \mathfrak{r}\varpi^4 + 2\mathfrak{s}\varpi^2\zeta^2 + \mathfrak{t}\zeta^4 + 2\mathfrak{u}\varpi^2 + 2\mathfrak{v}\zeta^2,$$

in which

$$\begin{array}{rl|l}
\mathfrak{R} = & \dfrac{\lambda}{a^2A} \times 64L^2 & + \dfrac{\lambda}{c^2C} \times 8Lm \\
\mathfrak{S} = & 48Lm & 4(4m^2+2LN) \\
\mathfrak{T} = & \frac{16}{3}(4m^2+2LN) & 40mN \\
\mathfrak{U} = & 16mN & 56N^2 \\
\mathfrak{r} = & 144pL & 2(8pm+4rL) \\
\mathfrak{s} = & 8(8pm+4rL) & 6(4pN+8rm) \\
\mathfrak{t} = & 4(4pN+8rm) & 120Nr \\
\mathfrak{u} = & 32p^2 & 8pr \\
\mathfrak{v} = & 16pr & 24r^2
\end{array}$$

We may now assume (*cf.* equation (133))

$$\int_0^\infty \left(DP - \tfrac{1}{8}fD^2P^2 + \tfrac{1}{192}f^2D^3P^2 - \tfrac{1}{9216}f^3D^4P^2\right)\frac{d\lambda}{\Delta}$$
$$= j_1(5w^6 - 90w^4z^2 + 120w^2z^4 - 16z^6) + j_2(3w^4 - 24w^2z^2 + 8z^4) + j_3(w^2 - 2z^2), \quad . \quad (137)$$

in which, on comparing coefficients, we obtain

$$5j_1 = \int_0^\infty \left[\frac{\mathfrak{R}}{A^6} - \frac{1}{8A^5}\left(\frac{\lambda}{a^2A}36\mathfrak{R} + \frac{\lambda}{c^2C}6\mathfrak{S}\right) + \frac{1}{192A^4}\left(\frac{\lambda^2}{a^4A^2}576\mathfrak{R} + \frac{\lambda^2}{a^2c^2AC}192\mathfrak{S} + \frac{\lambda^2}{c^4C^2}72\mathfrak{T}\right)\right.$$
$$\left. - \frac{1}{9216A^3}\left(\frac{\lambda^3}{a^6A^3}2304\mathfrak{R} + \frac{3\lambda^3}{a^4c^2A^2C}384\mathfrak{S} + \frac{3\lambda^3}{a^2c^4AC^2}288\mathfrak{T} + \frac{\lambda^3}{c^3C^3}720\mathfrak{U}\right)\right]\frac{d\lambda}{\Delta}.$$
$$. \quad . \quad . \quad . \quad (138)$$

There are corresponding equations for $3j_2$ and j_3, but these are not written down as they can readily be obtained from equations (135) and (136). To change $3h_2$ into $3j_2$, change all accented coefficients into black letter type, and change λ, λ^2, λ^3 wherever they occur explicitly, into $\frac{1}{2}\lambda$, $\frac{1}{3}\lambda^2$, $\frac{1}{4}\lambda^3$ respectively. The same procedure changes h_3 into j_3.

The values of these coefficients at the ellipsoidal point of bifurcation can be calculated from the material already provided in § 18 and the values of p and r given in formulæ (113). Using the exact solution throughout, I find

$$5j_1 = 0{\cdot}01931\,(\gamma-2) - 0{\cdot}01473,$$
$$3j_2 = 0{\cdot}01080\,(\gamma-2) + 0{\cdot}00576,$$
$$j_3 = 0{\cdot}5715\,\ (\gamma-2) + 0{\cdot}4094,$$

all terms in $(\gamma-2)^2$ vanishing in accordance with the result of § 14.

36. We turn now to the evaluation of ΔE_i. The potential of the whole mass at an internal point, V_i, is given, as in equation (20), by

$$V_i = \rho_0 (V_i(1) + \epsilon \Delta V_i(1) + \ldots) - (\rho_0 - \sigma)(E_i + \epsilon \Delta E_i + \ldots).$$

We have also

$$\nabla^2 V_i = -4\pi\rho = -4\pi\rho_0 + 4\pi(\rho_0 - \sigma)\left(\Sigma \frac{x^2}{a^2} + \epsilon P_0 + \epsilon^2 Q_0 + \ldots\right),$$

whence, by comparison

$$\nabla^2(\Delta E_i) = -4\pi P_0 = -4\pi\left(\frac{L}{a^8}w^4 + \frac{2m}{a^4c^4}w^2z^2 + \frac{N}{c^8}z^4 + \frac{2p}{a^4}w^2 + \frac{2r}{c^4}z^2\right). \quad (139)$$

We may assume

$$\Delta E_i = -4\pi[k_1w^6 + 3k_2w^4z^2 + 3k_3w^2z^4 + k_4z^6 + k_5w^4 + 2k_6w^2z^2 + k_7z^4 + 2k_8w^2 + 2k_9z^2],$$

in which equation (139) necessitates the following relations between the coefficients

$$\left\{\begin{aligned} 36k_1 + 6k_2 &= \frac{L}{a^8}, \\ 48k_2 + 36k_3 &= \frac{2m}{a^4c^4}, \\ 12k_3 + 30k_4 &= \frac{N}{c^8}, \end{aligned}\right. \qquad \left\{\begin{aligned} 16k_5 + 4k_6 &= \frac{2p}{a^4}, \\ 8k_6 + 12k_7 &= \frac{2r}{c^4}, \end{aligned}\right. \qquad 2k_8 + k_9 = 0.$$

By comparison with equation (123)

$$k_1 = \tfrac{1}{4}abc\left[LJ_{A^5} - \frac{2L}{a^2}I_{A^5} - \frac{m}{2c^2}I_{A^4C} + \tfrac{1}{192}\left(64\frac{L}{a^4}H_{A^5} + 32\frac{m}{a^2c^2}H_{A^4C} + 24\frac{N}{c^4}H_{A^3C^2}\right)\right],$$

$$k_5 = \tfrac{1}{4}abc\left[2pJ_{A^3} - \frac{p}{a^2}I_{A^3} - \tfrac{1}{2}\frac{r}{c^2}I_{A^2C}\right],$$

$$2k_8 = \tfrac{1}{4}abc\left[\frac{2L}{a^2}I_{A^3} + \tfrac{1}{2}\frac{m}{c^2}I_{A^2C} - \tfrac{1}{64}\left(64\frac{L}{a^4}H_{A^3} + 32\frac{m}{a^2c^2}H_{A^2C} + 24\frac{N}{c^4}H_{AC^2}\right)\right].$$

By direct computation from these values I find

$$\frac{4k_1}{abc} = 0{\cdot}019734(\gamma - 2) - 0{\cdot}021633,$$

$$\frac{4k_5}{abc} = -0{\cdot}007359,$$

$$\frac{8k_8}{abc} = -0{\cdot}017022.$$

37. We now return to equation (126). The solution has been assumed to be

$$Q_0 = Q'_0 + (\gamma-2)\,Q''_0,$$

in which $(\gamma-2)\,Q''_0$ is the contribution from the terms on the right

$$\theta(\gamma-2)\left[\tfrac{1}{6}(\gamma-3)\left(\Sigma\frac{x^2}{a^2}\right)^3-(\gamma-2)\,P''_0\left(\Sigma\frac{x^2}{a^2}\right)\right].$$

It follows that Q'_0 must satisfy the equation

$$\int_0^\infty [Q'-\tfrac{1}{4}f\,DQ'+\tfrac{1}{64}f^2\,D^2Q'-\tfrac{1}{2304}f^3\,D^3Q'+\ldots]\frac{d\lambda}{\Delta}-\theta Q'_0$$

$$= -\theta(\gamma-2)\,P'_0\left(\Sigma\frac{x^2}{a^2}\right)+\tfrac{1}{8}\int_0^\infty[DP^2-\tfrac{1}{8}f\,D^2P^2+\ldots]\frac{d\lambda}{\Delta}$$

$$-\frac{\Delta E_i}{\pi abc}+\delta n\,(x^2+y^2).$$

Substituting the values which have been assumed for the various terms in this equation, it becomes

$$h_1(5w^6-90w^4z^2+120w^2z^4-16z^6)+h_2(3w^4-24w^2z^2+8z^4)+h_3(w^2-2z^2)$$

$$-\theta\left[\frac{R'}{a^{12}}w^6+\frac{3S'}{a^8c^4}w^4z^2+\frac{3T'}{a^4c^8}w^2z^4+\frac{U'}{c^{12}}z^6+\frac{r'}{a^8}w^4+\frac{2s'}{a^4c^4}w^2z^2+\frac{t'}{c^8}z^4+\frac{2u'}{a^4}w^2+\frac{2v'}{c^4}z^2\right]$$

$$= -\theta(\gamma-2)\left(\frac{w^2}{a^2}+\frac{z^2}{c^2}\right)\left(\frac{L'}{a^8}w^4+\frac{2m'}{a^4c^4}w^2z^2+\frac{N'}{c^8}z^4+\frac{2p}{a^4}w^2+\frac{2r}{c^4}z^2\right)$$

$$+\tfrac{1}{8}[j_1(5w^6-90w^4z^2+120w^2z^4-16z^6)+j_2(3w^4-24w^2z^2+8z^4)+j_3(w^2-2z^2)]$$

$$+\frac{4}{abc}[k_1w^6+3k_2w^4z^2+3k_3w^2z^4+k_4z^6+k_5w^4+2k_6w^2z^2+k_7z^4+2k_8w^2+2k_9z^2]$$

$$+\delta n\,(x^2+y^2). \quad \ldots \quad (140)$$

Equating coefficients of w^6, w^4z^2, w^2z^4 and z^6, we find

$$5h_1-\theta\left[\frac{R'-(\gamma-2)\,L'a^2}{a^{12}}\right]=\tfrac{5}{8}j_1+\frac{4k_1}{abc}, \quad \ldots \quad (141)$$

$$-90h_1-\theta\left[\frac{3S'-(\gamma-2)(2m'a^2+L'c^2)}{a^8c^4}\right]=-\tfrac{90}{8}j_1+\frac{12k_2}{abc}, \quad \ldots \quad (142)$$

$$120h_1-\theta\left[\frac{3T'-(\gamma-2)(N'a^2+2m'c^2)}{a^4c^8}\right]=120j_1+\frac{12k_3}{abc}, \quad \ldots \quad (143)$$

$$-16h_1-\theta\left[\frac{U'-(\gamma-2)\,N'c^2)}{a^{12}}\right]=-\tfrac{16}{8}j_1+\frac{4k_4}{abc}. \quad \ldots \quad (144)$$

On multiplying equations (141) and (142) by 18, 1, and adding, we obtain, with the help of the relations of § 36,

$$18\left[\frac{R'-(\gamma-2)\,L'a^2}{a^{12}}\right]+\left[\frac{3S'-(\gamma-2)\,(2m'a^2+L'c^2)}{a^8c^4}\right]=-\frac{2}{\theta}\frac{L}{a^8}. \quad . \quad . \quad (145)$$

Equations (142) to (144), treated in a similar manner, give

$$4\left[\frac{3S'-(\gamma-2)\,(2m'a^2+L'c^2)}{a^8c^4}\right]+3\left[\frac{3T'-(\gamma-2)\,(N'a^2+2m'c^2)}{a^4c^8}\right]=-\frac{2}{\theta}\frac{m}{a^4c^4}, \quad (146)$$

$$2\left[\frac{3T'-(\gamma-2)\,(N'a^2+2m'c^2)}{a^4c^8}\right]+15\left[\frac{U'-(\gamma-2)\,N'c^2}{c^{12}}\right]=-\frac{2}{\theta}\frac{N}{c^8}. \quad . \quad . \quad (147)$$

These equations, taken with one other, suffice to determine R′, S′, T′, U′. For additional equation we shall use equation (141).

The values of j_1 and k_1 have already been given. For $5h_1$ I find, by direct computation from formula (134),

$$5h_1 = 0{\cdot}00152R'-0{\cdot}01345S'+0{\cdot}03926T'-0{\cdot}03813U',$$

so that equation (141) reduces to

$$-\theta\left[\frac{R'-(\gamma-2)\,L'a^2}{a^{12}}\right]+0{\cdot}00152R'-0{\cdot}01345S'+0{\cdot}03926T'-0{\cdot}03183U'$$
$$=0{\cdot}02214\,(\gamma-2)-0{\cdot}02347. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (148)$$

Solving the system of four equations (145) to (148), I find

$$\frac{R'-(\gamma-2)\,L'a^2}{a^{12}}=0{\cdot}04794-0{\cdot}04309\,(\gamma-2),$$

$$\frac{3S'-(\gamma-2)\,(2m'a^2+N'c^2)}{a^8c^4}=0{\cdot}1894-0{\cdot}2573\,(\gamma-2),$$

$$\frac{3T'-(\gamma-2)\,(N'a^2+2m'c^2)}{a^4c^8}=0{\cdot}4388-0{\cdot}6707\,(\gamma-2),$$

$$\frac{U'-(\gamma-2)\,N'c^2}{c^{12}}=0{\cdot}2605-0{\cdot}5077\,(\gamma-2),$$

leading to the values

$$\left.\begin{aligned}R' &= 0{\cdot}4155\ \ -1{\cdot}7799\,(\gamma-2),\\ 3S' &= 0{\cdot}1894\ \ -1{\cdot}4585\,(\gamma-2),\\ 3T' &= 0{\cdot}0506\ \ -0{\cdot}4124\,(\gamma-2),\\ U' &= 0{\cdot}00346-0{\cdot}0375\,(\gamma-2).\end{aligned}\right\}\text{(exact solution)}. \quad . \quad . \quad . \quad . \quad (149)$$

This may be compared with the approximation expressed by equation (130), which gives

$$\left.\begin{aligned}
R' &= (\gamma-2)\,L'a^2 - \frac{4}{abc}\,k_1\,\frac{a^{12}}{\theta} &&= 0{\cdot}3981 \;\; -1{\cdot}7691\,(\gamma-2),\\
3S' &= (\gamma-2)\,(2m'a^2+L'c^2) - \frac{4}{abc}\,3k_2\,\frac{a^8c^4}{\theta} &&= 0{\cdot}2013 \;\; -1{\cdot}4659\,(\gamma-2),\\
3T' &= (\gamma-2)\,(N'a^2+2m'c^2) - \frac{4}{abc}\,3k_3\,\frac{a^4c^8}{\theta} &&= 0{\cdot}0452 \;\; -0{\cdot}4090\,(\gamma-2),\\
U' &= (\gamma-2)\,N'c^2 - \frac{4}{abc}\,k_4\,\frac{c^{12}}{\theta} &&= 0{\cdot}00355 - 0{\cdot}0381\,(\gamma-2).
\end{aligned}\right\} \text{(Approximation A)} \quad . \quad (150)$$

The error of Approximation A is now something like 5 per cent. in the terms independent of $(\gamma-2)$, and there is also an additional small error of less than 1 per cent. in the terms containing $(\gamma-2)$.

38. We pass now to the discussion of terms of degree 4. Equating coefficients of w^4, w^2z^2, and z^4 in equation (140) gives

$$\left.\begin{aligned}
3h_2 - \theta\left[\frac{\mathrm{r}'-2\,(\gamma-2)\,pa^2}{a^8}\right] &= \frac{3j_2}{8} + \frac{4k_5}{abc},\\
-24h_2 - \theta\left[\frac{2\mathrm{s}'-2\,(\gamma-2)\,(ra^2+pc^2)}{a^4c^4}\right] &= -\frac{24j_2}{8} + \frac{8k_6}{abc},\\
8h_2 - \theta\left[\frac{\mathrm{t}'-2\,(\gamma-2)\,rc^2}{c^8}\right] &= \frac{8j_2}{8} + \frac{4k_7}{abc}.
\end{aligned}\right\} \quad . \quad . \quad . \quad (151)$$

Following the method of the last section, the last two equations may be replaced by

$$8\left[\frac{\mathrm{r}'-2\,(\gamma-2)\,pa^2}{a^8}\right] + \left[\frac{2\mathrm{s}'-2\,(\gamma-2)\,(ra^2+pc^2)}{a^4c^4}\right] = -\frac{2}{\theta}\frac{2p}{a^4}, \quad . \quad . \quad . \quad (152)$$

$$\left[\frac{2\mathrm{s}'-2\,(\gamma-2)\,(ra^2+pc^2)}{a^4c^4}\right] + 3\left[\frac{\mathrm{t}'-(\gamma-2)\,rc^2}{c^8}\right] = -\frac{2}{\theta}\frac{r}{c^4}. \quad . \quad . \quad . \quad (153)$$

The three equations (151) to (153) determine r′, s′ and t′. The values of $4k_5/abc$ and of $3j_2$ have already been given. For $3h_2$ I obtain by direct computation from formula (135)

$$3h_2 = 0{\cdot}010972\mathrm{r}' - 0{\cdot}064324\mathrm{s}' + 0{\cdot}094925\mathrm{t}'$$
$$+ 0{\cdot}01031R' - 0{\cdot}007283S' - 0{\cdot}04422T' + 0{\cdot}32343U',$$

which, on inserting the exact solution (149) for R′, S′, T′, U′, becomes

$$3h_2 = 0{\cdot}010972\mathrm{r}' - 0{\cdot}064324\mathrm{s}' + 0{\cdot}094925\mathrm{t}' + 0{\cdot}00179 - 0{\cdot}00166\,(\gamma-2).$$

The smallness of the last two terms of course measures the closeness of the approximation (130).

Equation (151) now reduces to

$$-\theta\left[\frac{\mathrm{r}'-2(\gamma-2)pa^2}{a^8}\right]+0{\cdot}010972\mathrm{r}'-0{\cdot}064324\mathrm{s}'+0{\cdot}094925\mathrm{t}'$$
$$= -0{\cdot}00843+0{\cdot}00310(\gamma-2). \quad . \quad . \quad (154)$$

On solving equations (152) to (154) directly, I find

$$\frac{\mathrm{r}'-2(\gamma-2)pa^2}{a^8} = 0{\cdot}02074-0{\cdot}01055(\gamma-2),$$

$$\frac{2\mathrm{s}'-2(\gamma-2)(ra^2+pc^2)}{a^4c^4} = -0{\cdot}07096+0{\cdot}08440(\gamma-2),$$

$$\frac{\mathrm{t}'-2(\gamma-2)rc^2}{c^8} = -0{\cdot}14008-0{\cdot}02813(\gamma-2),$$

whence the values of r′, s′, t′ are

$$\left.\begin{aligned}\mathrm{r}' &= 0{\cdot}08755-0{\cdot}10962(\gamma-2),\\ \mathrm{s}' &= -0{\cdot}01727+0{\cdot}04871(\gamma-2),\\ \mathrm{t}' &= -0{\cdot}007862+0{\cdot}02511(\gamma-2).\end{aligned}\right\}\text{(exact solution)}. \quad . \quad . \quad . \quad . \quad (155)$$

This may be compared with Approximation A, namely,

$$\mathrm{r}' = 2(\gamma-2)pa^2-\frac{4k_5}{abc}\frac{a^8}{\theta}, \text{ \&c.,}$$

which leads to the approximate values

$$\left.\begin{aligned}\mathrm{r}' &= 0{\cdot}06590-0{\cdot}06509(\gamma-2),\\ \mathrm{s}' &= -0{\cdot}00728+0{\cdot}02815(\gamma-2),\\ \mathrm{t}' &= -0{\cdot}007929+0{\cdot}02669(\gamma-2).\end{aligned}\right\}\text{(Approximation A)}.$$

The percentage error is large, although the absolute error in the coefficients is fairly small. This will be readily understood on noticing that under Approximation B, r′, s′ and t′ would vanish altogether.

39. Finally equating coefficients of w^2 and z^2 in equation (140) and making use of the relation $k_9 = -2k_8$, we obtain

$$h_3-\frac{2\mathrm{u}'}{a^4}\theta = \tfrac{1}{8}j_3+\frac{8k_8}{abc}+\delta n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (156)$$

$$-2h_3-\frac{2\mathrm{v}'}{c^4}\theta = -\tfrac{1}{4}j_3-\frac{16k_8}{abc}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (157)$$

Multiplying by 2, 1 and adding,

$$\delta n = -2\theta\left(\frac{2u'}{a^4} + \frac{v'}{c^4}\right), \quad \ldots \ldots \ldots \quad (158)$$

an equation which may be used to replace (156).

By direct computation from equation (136), I find

$$\begin{aligned} h_3 &= 0{\cdot}19769u' - 0{\cdot}58215v' \\ &\quad + 0{\cdot}06706r' + 0{\cdot}01164s' - 0{\cdot}61497t' \\ &\quad + 0{\cdot}04270R' + 0{\cdot}03778S' - 0{\cdot}07530T' - 1{\cdot}1894U' \\ &= 0{\cdot}19769u' - 0{\cdot}58215v' + 0{\cdot}02348 - 0{\cdot}11113(\gamma - 2), \end{aligned}$$

so that equation (157) reduces to

$$0{\cdot}39538u' + 2{\cdot}8142v' = 0{\cdot}0212 + 0{\cdot}0794(\gamma - 2). \quad \ldots \ldots \quad (159)$$

Equations (158) and (159) are adequate to give u′ and v′ in terms of any assigned value of δn. We require especially the solution at the actual point of bifurcation of the compressible series, and to obtain this the two equations must be combined with a third equation expressing the condition for a point of bifurcation.

The third equation is

$$\delta(a^6 c J_{AA}) = 0,$$

or, transforming by the method of § 27,

$$\frac{6u'}{a^2} I_{AAA} + \frac{v'}{c^2} I_{AAC} = \frac{24p^2}{a^4} H_{A^4} + \frac{6pr}{a^2c^2} H_{A^3C} + \frac{3r^2}{2c^4} H_{A^2C^2}.$$

Inserting numerical values, this becomes

$$0{\cdot}24919u' + 0{\cdot}20878v' = 0{\cdot}00024. \quad \ldots \ldots \ldots \quad (160)$$

From this and equation (159), I found by direct solution

$$\left.\begin{aligned} u' &= -0{\cdot}00550 - 0{\cdot}02651(\gamma - 2) \\ v' &= 0{\cdot}00778 + 0{\cdot}03195(\gamma - 2) \end{aligned}\right\}, \quad \ldots \ldots \quad (161)$$

and equation (158) now gives

$$\delta n = -0{\cdot}01292 - 0{\cdot}05495(\gamma - 2). \quad \ldots \ldots \ldots \quad (162)$$

The solution which has just been completed, combined with the two exact solutions (127) and (128) previously known, will give the exact solution for a compressible mass at the point of bifurcation at which the pseudo-spheroidal series gives up its stability to the pseudo-ellipsoidal series.

40. The rotation at this point of bifurcation is given by

$$\frac{\omega^2}{2\pi\rho_0} = n+\epsilon\,\Delta n+\epsilon^2\,\delta n$$

$$= 0{\cdot}18712-0{\cdot}04400\left(\frac{\rho_0-\sigma}{\rho_0}\right)-[0{\cdot}01292+0{\cdot}05495\,(\gamma-2)]\left(\frac{\rho_0-\sigma}{\rho_0}\right)^2. \quad (163)$$

The first point of interest about this equation is that the term in $(\rho_0-\sigma)$ is independent of γ. This must necessarily be the case, for we have seen that for an incompressible mass, regarded as a special case of the compressible mass that has been under discussion, $\gamma-2$ is infinite while $\rho_0-\sigma$ vanishes, and the product of the two remains finite. Thus for an incompressible mass (163) reduces to

$$\omega^2/2\pi\rho_0 = 0{\cdot}18712,$$

which is the true value, but if there has been a term in $(\gamma-2)$ multiplying $(\rho_0-\sigma)$, equation (163) would have led to a wrong value.

When $\gamma = 2$, equation (163) reduces to

$$\omega^2/2\pi\rho_0 = 0{\cdot}18712-0{\cdot}04400\epsilon-0{\cdot}01292\epsilon^2-\ldots.$$

Although we cannot be perfectly sure, the series is almost certainly convergent right up to the limiting case of $\sigma = 0$ or $\epsilon = 1$. In this case, it appears to converge to a limit of about $\omega^2/2\pi\rho_0 = 0{\cdot}120$, but unfortunately it is impossible to evaluate the limit for other values of γ.

A more important question is the relation between ω^2 and $\bar{\rho}$ at the point of bifurcation.

We have evaluated ρ as far as ϵ^3, but for comparison with the value of ω^2 given by equation (163), it will be enough to evaluate $\bar{\rho}$ as far as ϵ^2. Furthermore, to avoid a very complicated integration, we shall use Approximation B for the terms in ϵ^2, and so take

$$\rho = \rho_0-(\rho_0-\sigma)\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+\epsilon\eta\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^2\right],$$

where $\eta = \frac{1}{2}(\gamma-2{\cdot}75)$, this being the approximation given by equation (84). By a simple projective transformation, it follows that the mean density is the same as that in a sphere in which the law of density is

$$\rho = \rho_0-(\rho_0-\sigma)\left(\frac{r^2}{a^2}+\epsilon\eta\frac{r^4}{a^4}\right).$$

As far as terms in ϵ, the radius of this sphere is given by $r_0 = (1-\frac{1}{2}\epsilon\eta)\,a$, and the mass taken as far as ϵ^2 is

$$4\pi\int_0^{r_0}\rho r^2\,dr = 4\pi\left[\frac{r_0^{\,3}}{3}\,\rho_0-(\rho_0-\sigma)\left(\frac{r_0^{\,5}}{5a^2}+\epsilon\eta\frac{r_0^{\,7}}{7a^2}\right)\right],$$

whence the mean density, as far as ϵ^2, is found to be

$$\bar{\rho} = \rho_0\left(1-\tfrac{3}{5}\epsilon+\tfrac{6}{35}\epsilon^2\eta\right) = \rho_0\left[1-\tfrac{3}{5}\epsilon-\tfrac{9}{140}\epsilon^2-\ldots+\tfrac{3}{35}(\gamma-2)\,\epsilon^2+\ldots\right]. \quad . \quad . \qquad (164)$$

From these equations we obtain

$$\frac{\omega^2}{2\pi\bar{\rho}} = 0{\cdot}18712+0{\cdot}06827\epsilon+(0{\cdot}01602+0{\cdot}07098\,(\gamma-2))\,\epsilon^2+\ldots. \quad . \quad . \qquad (165)$$

Ratio of Centrifugal Force to Gravity.

41. [§ 41 *revised August* 31, 1918].—The ratio of centrifugal force to gravity is interesting only when it becomes equal to unity. When this occurs a stream of matter is ejected at the points at which centrifugal force equals gravity. Centrifugal force will first become equal to gravity at points which are furthest from the axis of rotation, and these will be points on the equator of the rotating mass. Considering a point on the axis of x, the condition for centrifugal force to be equal to gravity is

$$\partial \mathrm{V}/\partial x+\omega^2 x = 0,$$

or

$$\partial \Omega/\partial x = 0.$$

Since, by equation (30), $\Omega = \dfrac{\kappa\gamma}{\gamma-1}\,\rho^{\gamma-1}+a$ *constant*, this may equally well be expressed in the form

$$\partial \rho/\partial x = 0.$$

The general value of ρ is given by equation (45). On the x-axis this reduces to

$$\rho = \rho_0\,(1-\epsilon \mathrm{F}),$$

where

$$\mathrm{F} = \frac{x^2}{a^2}+\epsilon\left[\frac{\mathrm{L}x^4}{a^8}+\frac{2px^2}{a^4}\right]+\epsilon^2\left[\frac{\mathrm{R}x^6}{a^{12}}+\frac{\mathrm{r}x^4}{a^8}+\frac{2\mathrm{u}x^2}{a^2}\right]+\ldots. \quad . \quad . \quad . \qquad (166)$$

The intercepts on the axis of x are determined by the condition $\rho = \sigma$, and so are given by $\mathrm{F} = 1$. The solution of this equation is found to be

$$\frac{x^2}{a^2} = 1-\epsilon\left(\frac{\mathrm{L}}{a^4}+\frac{2p}{a^2}\right)-\epsilon^2\left[\frac{\mathrm{R}}{a^6}+\frac{\mathrm{r}}{a^4}+\frac{2\mathrm{u}}{a^2}-\left(\frac{\mathrm{L}}{a^4}+\frac{2p}{a^2}\right)\left(\frac{2\mathrm{L}}{a^4}+\frac{2p}{a^2}\right)\right]+\ldots. \quad . \qquad (167)$$

The points on the x-axis at which $\partial\rho/\partial x = 0$ are given by $\partial \mathrm{F}/\partial x = 0$, so that it appears that $\partial\rho/\partial x$ will just vanish at the extremities of the x-axis if $\partial \mathrm{F}/\partial x$ vanishes for the value of x given in equation (167).

The equation $\partial \mathrm{F}/\partial x = 0$ becomes, on division by $2x/a^2$

$$1+\epsilon\left[\frac{2\mathrm{L}}{a^4}\left(\frac{x^2}{a^2}\right)+\frac{2p}{a^2}\right]+\epsilon^2\left[\frac{3\mathrm{R}}{a^6}\left(\frac{x^4}{a^4}\right)+\frac{2\mathrm{r}}{a^4}\left(\frac{x^2}{a^2}\right)+\frac{2\mathrm{u}}{a^2}\right]+\ldots = 0,$$

and, on inserting the value of x^2/a^2 given by equation (167), this becomes

$$1+\epsilon\left[\frac{2L}{a^4}+\frac{2p}{a^2}\right]+\epsilon^2\left[\frac{3R}{a^6}+\frac{2r}{a^4}+\frac{2u}{a^2}-\frac{2L}{a^4}\left(\frac{L}{a^4}+\frac{2p}{a^2}\right)\right]+\ldots = 0.$$

42. Let us examine in particular the special form assumed by this equation for the special configuration at which the pseudo-spheroidal form becomes unstable, giving place to a pseudo-ellipsoidal form. Inserting the numerical values for L, p, R, &c., obtained in §§ 18–39, the equation becomes

$$1+\epsilon[(\gamma-2)-1\cdot0509]+\epsilon^2[\tfrac{1}{2}(\gamma-2)^2-0\cdot4063(\gamma-2)-0\cdot0510]+\ldots = 0. \quad (168)$$

For a given value of γ this equation determines the value of ϵ for which centrifugal force just outbalances gravity as the pseudo-spheroidal form gives place to the pseudo-ellipsoidal.

For instance, for the value $\gamma = 2$, the equation becomes

$$1-1\cdot0509\epsilon-0\cdot0510\epsilon^2-\ldots.$$

The values of ϵ obtained by using terms as far as ϵ and ϵ^2 respectively are 0·9516 and 0·9112. The true root is perhaps somewhere near $\epsilon = \frac{9}{10}$. Thus a mass of rotating matter obeying LAPLACE'S law ($\gamma = 2$) will throw off matter from its equator before reaching the ellipsoidal point of bifurcation if $\epsilon > \frac{9}{10}$, or if σ is less than $\frac{1}{10}\rho$.

The root $\epsilon = \frac{9}{10}$ agrees well with the corresponding quantity in the two-dimensional problem. For cylindrical masses obeying LAPLACE'S law ($\gamma = 2$) the problem can be solved exactly and the root is found to be $\epsilon = 1$, giving $\sigma = 0$.*

Equation (168) may more usefully be regarded as giving γ when the value of ϵ is assigned. The only value of ϵ which is of any astronomical interest is the value $\epsilon = 1$, the value for a mass whose density reduces to zero at the boundary. Putting $\epsilon = 1$, we obtain an equation giving a critical value of γ.

From terms as far as ϵ only, the value of γ is clearly enough

$$\gamma = 2\cdot0509.$$

On including terms as far as γ^2, this value becomes

$$\gamma = 2\cdot1521.$$

These values of γ appear to be converging to a limit; we cannot state it with great accuracy, but we shall perhaps not be far wrong if we assume it to be $\gamma = 2\cdot2$.

Here again we may compare the problem with the simpler two-dimensional one in which, as follows from what has already been said, the root of $F = 0$, when $\epsilon = 1$, is $\gamma = 2$ exactly.

* 'Phil. Trans.,' A, vol. 213, p. 471.

The Influence of other Physical Factors.

43. The discussion from § 10 onwards has been limited to material in which the relation between pressure and density is of the form

$$p = \kappa\rho^{\gamma} - \text{cons.}, \qquad (169)$$

an equation which implies that the mass is composed of matter of similar properties throughout. In an actual mass there must be at least some tendency for the substances of greater atomic or molecular weight to sink towards the centre, although it is far from clear to what extent this tendency will actually prevail under astronomical conditions.

In equation (169), γ is usually regarded as expressing the measure of a property of the material, being the ratio of the specific heats when the matter is gaseous. We have, however, only used this equation in its differential form

$$dp = \kappa\gamma\rho^{\gamma-1}\, d\rho. \qquad (170)$$

Now with any structure whatever of the mass of matter under consideration the surfaces of constant pressure and density must coincide. Our use of equation (170) has in no sense referred to a change of state of a single element of matter, it has referred to a passage from one surface of constant density to the next. Thus, in strictness, γ refers to the arrangement of the matter and not to a physical property of the matter. In the case so far considered in which the matter has been supposed to have the same physical properties throughout, the two meanings become identical, but when the mass under discussion is an aggregate of imperfectly mixed types of matter, it becomes important to distinguish clearly between the two meanings.

When p and ρ both vanish at the surface of the mass, equation (170) is equivalent to

$$d\log\rho = \frac{1}{\gamma}\, d\log p, \qquad (171)$$

where the differentials are supposed to refer to passage from one surface of constant density to the next. In passing from one surface to the next, ρ will increase partly on account of the increase of pressure, but partly also on account of the higher atomic weight of which the materials of the layer of higher density may be supposed composed. Thus we may suppose equation (171) replaced by

$$d\log\rho = \left(\frac{1}{\gamma_{\mathrm{M}}} + \frac{1}{\gamma_{\mathrm{A}}}\right) d\log p, \qquad (172)$$

where γ_{M} is the value of γ defined by equation (169) as a property of the material, while γ_{A} is such that the additional term on the right of equation (172) represents the increase in $d\log\rho$ owing to increase of atomic weight of the materials as we pass to layers of greater density.

In any arrangement of matter whatever, γ_A will have some value for each layer of constant density, being virtually defined by equation (172). There is no reason to expect that γ_A will be constant, or even approximately constant throughout the mass. But in the present complex problem we must be content to discover tendencies rather than exact results, and so may think of γ_A as a constant. An increase of atomic weight on passing to layers of greater density will give a positive value to γ_A, while similarly a diminution of atomic weight would give a negative value of γ_A.

The whole of the foregoing analysis will now apply to the present case if we put

$$\frac{1}{\gamma} = \frac{1}{\gamma_M} + \frac{1}{\gamma_A}. \qquad (173)$$

It has been seen that for values of γ less than a critical value, which may be taken (although only as a rough approximation) to be 2·2, it will be impossible for the rotating mass to assume the ellipsoidal form, and so impossible for it to separate into two detached masses. The condition for assuming the ellipsoidal form now becomes

$$\frac{1}{\gamma_M} + \frac{1}{\gamma_A} < \frac{1}{2{\cdot}2}. \qquad (174)$$

Thus, if γ_A is positive (atomic weight increasing towards centre), the critical value of γ_M will be greater than 2·2; with γ_A negative, the critical value of γ_M will be less than 2·2. It is possible for a heterogeneous but perfectly incompressible mass ($\gamma_M = \infty$) to fail to attain the ellipsoidal shape if γ_A is small enough—*i.e.*, if the layers increase sufficiently rapidly in density from increase of atomic weight alone.

44. In the foregoing discussion we have considered only the effects of continuous changes of atomic weight. Entirely confirmatory results may be obtained from a consideration of excessively abrupt changes.

Let us consider only the simplest case in which the mass is formed of completely separated layers of two substances only, one very light and the other very heavy—to make the picture definite, let us think of an inner mass of iron vapour surrounded by an atmosphere of hydrogen, and let us suppose the density of the hydrogen may be neglected in comparison with that of the iron, so that the gravitational field will be the same as if the iron alone were present.

For any given value of ω we can draw the equipotentials $\Omega =$ cons., and the boundary of the mass of iron must be one of these, say $\Omega = C_1$. Proceeding outwards and drawing the exterior equipotentials we must in time come to one having a double point, say $\Omega = C_x$, and at the double point on this equipotential centrifugal force will be exactly equal to gravity ($\partial\Omega/\partial n = 0$). The space between the equipotentials $\Omega = C_1$ and $\Omega = C_x$ can be filled with hydrogen without matter being thrown off by rotation, but clearly no greater volume of hydrogen than this can be retained.

As the rotation ω increases from zero upwards, the equipotential $\Omega = C_x$ starts from infinity and moves inwards, so that the maximum volume of hydrogen which

can be retained continually decreases. With an allotted amount of hydrogen, the ellipsoidal point of bifurcation may or may not be reached before matter has begun to be ejected from the equator of the figure.

The simplest case for accurate discussion occurs when the core is treated as incompressible. In fig. 3 the thick curve represents the cross-section through the axis of rotation of an incompressible mass at the Maclaurin-Jacobian point of bifurcation, while the thin curve represents the equipotential $\Omega = C_x$, which has a series of double points round its equator. If the volume of light material surrounding the core is just equal to the volume between these surfaces, matter will begin to be thrown off at the equator precisely at the moment at which the ellipsoidal point of bifurcation is reached. If the volume of light matter is at first

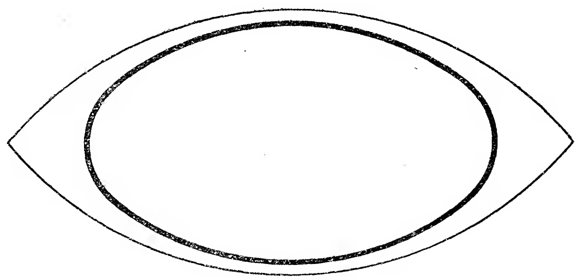

Fig. 3.

greater than this critical volume, matter will be thrown off equatorially before the ellipsoidal point of bifurcation is reached, the amount ejected being such that the volume is just reduced to the critical volume when this point is reached. Conversely, of course, if the volume of light matter is initially less than this, the point of bifurcation will be reached before any matter is thrown off equatorially.

A rough estimate shows that the critical volume, when the matter is incompressible, is about one-third of the volume of the core. For a compressible core for which $\gamma > 2{\cdot}2$, it is of course less, becoming equal to zero when $\gamma = 2{\cdot}2$ (approximately).

This completes our collection of theoretical results. They may now be recapitulated and discussed with reference to actual astronomical conditions.

Summary and Discussion of Results.

45. Our discussion began with a general survey of the types of configurations which can be assumed by compressible astronomical masses in rotation. The result announced in a previous paper* that the incompressible mass provides a good model from which to study the behaviour of compressible masses has on the whole been fully confirmed. A mass of incompressible matter, shrinking while rotating, will assume first the shape of a spheroid, then that of an ellipsoid, afterwards becoming unstable

* 'Phil. Trans.,' A, vol. 213, p. 457.

and probably dividing into two detached masses after passing through a series of pear-shaped configurations. In the present investigation it has been found that, except for an alternative possibility to be discussed later, a shrinking compressible mass will experience a very similar sequence of changes. It will pass through a sequence of figures which from their similarity to the spheroids and ellipsoids of incompressible masses, may be described as pseudo-spheroids and pseudo-ellipsoids. The pseudo-spheroids become unstable when a certain degree of flatness is reached and give place to a series of pseudo-ellipsoids, these in turn become unstable when a certain degree of elongation is reached and give place to a series of pear-shaped figures which probably end by fission into detached masses.

46. It may simplify the presentation of the detailed results obtained, if we confine our attention at first to the innermost strata, this being supposed to mean strata so near the centre (which is taken as origin) that terms in x^4, x^2y^2, &c., may be neglected in comparison with terms in x^2, y^2, z^2. It has been found that these strata will be first truly spheroidal, then truly ellipsoidal, then pear-shaped, each change of figure corresponding, as in the incompressible mass, to a passage through a point of bifurcation of the system as a whole.

The problem has been studied in detail for a pressure-density law of the form

$$p = \kappa\rho^{\gamma} - \text{cons.}, \qquad (175)$$

and special attention has been paid to the exact position of the configuration at which pseudo-spheroidal configurations give place to pseudo-ellipsoidal ones. This configuration we may refer to as the ellipsoidal point of bifurcation; its special importance will appear later. The corresponding configuration for an incompressible mass is the Maclaurin-Jacobian point of bifurcation; at this point the semi-axes a_0, c_0 and the angular velocity ω_0 are given by

$$a_0 = 1{\cdot}1972, \qquad c_0 = 0{\cdot}69766, \qquad \omega_0^2/2\pi = 0{\cdot}18712\rho,$$

so that the equation of the boundary is

$$\frac{x^2+y^2}{a_0^2} + \frac{z^2}{c^2} = 1.$$

For a compressible mass, the equations of the innermost strata of constant density at the ellipsoidal point of bifurcation have been found to be

$$\frac{x^2+y^2}{a_0^2}[1-0{\cdot}016307\epsilon-(0{\cdot}00384+0{\cdot}01850\,(\gamma-2))\,\epsilon^2-\ldots]$$
$$+\frac{z^2}{c_0^2}[1+0{\cdot}05634\epsilon+(0{\cdot}01598+0{\cdot}06565\,(\gamma-2))\,\epsilon^2+\ldots] = \frac{\rho_0-\rho}{\rho_0-\sigma}, \qquad (176)$$

where ρ_0 is the density at the centre, σ that at the surface, and ϵ stands for $(\rho_0-\sigma)/\rho_0$. For the important case of a gaseous mass, ϵ is of course equal to unity.

Thus at the ellipsoidal point of bifurcation, the innermost strata of equal density are of flatter shape than those for an incompressible mass, showing that compressibility tends to postpone instability for the spheroidal mass.

The value of ω, the angular velocity at this point of bifurcation has been found to be given by

$$\omega^2/2\pi\rho_0 = 0{\cdot}18712-0{\cdot}04400\epsilon-(0{\cdot}01292+0{\cdot}05495\,(\gamma-2))\,\epsilon^2-\ldots, \quad . \quad . \quad (177)$$

or, if we evaluate ω^2 in terms of $\bar{\rho}$, the mean density of the whole mass,

$$\omega^2/2\pi\bar{\rho} = 0{\cdot}18712+0{\cdot}06827\epsilon+(0{\cdot}01602+0{\cdot}07098\,(\gamma-2))\,\epsilon^2-\ldots, \quad . \quad . \quad (178)$$

the coefficient of ϵ^2 now being only approximate. We notice that $\omega^2/2\pi\bar{\rho}$ is greater for a compressible mass than for an incompressible mass, so that again compressibility may be said to postpone the instability of the spheroidal form.*

Equation (176) applies only to the innermost strata for which x^4, &c., may be neglected. The equation of the outer strata, both at the point of bifurcation and elsewhere, are found to contain terms of degrees four and higher, there being terms of degrees four and two multiplied by ϵ, terms of degrees six, four and two multiplied by ϵ^2, and so on. The terms of degrees six and four have been calculated for the ellipsoidal point of bifurcation and the terms of degree four for the pear-shaped point of bifurcation.

The presence of these terms destroys the spheroidal or ellipsoidal shape of the outer strata. In general it is found that the outer strata are more lens-shaped than the inner strata when these latter are spheroidal, and more spindle-shaped than the inner strata when these latter are ellipsoidal. The lens-shaped form of the outer strata may go so far that the outer boundary develops a sharp edge. When this occurs, centrifugal force is exactly equal to gravity at points on the periphery of the lens, and any further increase in the rotation of the mass results in matter being thrown off from round this periphery. Similarly the spindle-shaped figure may develop sharp ends, in which case matter will be thrown off here also.

47. Consider now a gradually shrinking mass of gas or other compressible matter, the rotation increasing as the shrinkage proceeds. For a very slow rotation the strata and the boundary will all be spheroidal. As the rotation increases, the boundary departs more and more from the spheroidal form, taking a series of forms

* All terms in $\omega^2/2\pi\bar{\rho}$ are positive because $\gamma-2$ is necessarily positive; for, as we shall see, if $\gamma<2$, the compressibility so far postpones the occurrence of the ellipsoidal point of bifurcation that it does not occur at all. Incidentally equation (178) has an important bearing on the origin of the solar system. It shows that for every mass which has broken up by fission $\omega^2/2\pi\bar{\rho}$ must at some time have exceeded the value 0·18712. This provides the keystone, which has so far been wanting, in an argument I have given elsewhere ('M. N. Royal Ast. Soc.,' 77, p 191) to show that the solar system is very unlikely to have broken up by rotation alone.

which we have called pseudo-spheroidal, these being more lens-shaped than true spheroids. There are two alternatives. It may be that the point of bifurcation will be reached before a sharp edge forms at the equator of the pseudo-spheroids, and if this happens the innermost strata will become ellipsoidal, and the outer strata and the boundary will be pseudo-ellipsoidal; the mass will proceed towards the pear-shaped form and ultimate fission. But, instead of this happening, it may be that a sharp edge will be formed before the point of bifurcation is reached, and the mass will disintegrate through equatorial loss of matter. It is of the utmost importance to determine which of these events will happen first for a particular mass.

48. Some information may be obtained from a general survey of the problem. The spherical solutions for a mass of matter at rest obeying the law $p = \kappa\rho^{\gamma}$ have been investigated by RITTER, DARWIN, EMDEN* and others. Excepting the special case of $\gamma = 2$, in which the equation reduces to a linear equation, the solution can be expressed in finite terms in one case only, namely $\gamma = 1\frac{1}{5}$. In this case the solution, first given by SCHUSTER,† is

$$\rho = \mathrm{A}\,(1+r^2/a^2)^{-\frac{5}{2}}. \qquad (179)$$

This happens also to give the lowest value of γ for which the mass is finite. For values of γ less than $1\frac{1}{5}$, the matter extends to infinity and the total mass is infinite; when $\gamma = 1\frac{1}{5}$, the matter extends to infinity but the total mass is finite; when $\gamma > 1\frac{1}{5}$, the matter is of finite extent and of finite total mass. As γ increases the variations in density becomes less rapid and finally the value $\gamma = \infty$ corresponds to an incompressible mass in which the density is uniform throughout.

Now clearly a mass for which $\gamma = 1\frac{1}{5}$ will lose matter equatorially with even the slightest amount of rotation. For all except an infinitesimal fraction of the whole mass is concentrated in regions near the centre, and the potential near the edge may accordingly be taken to be M/r. The value of Ω is therefore

$$\Omega = \frac{\mathrm{M}}{r} + \tfrac{1}{2}\omega^2(x^2+y^2).$$

The problem is now seen to be identical with one which has been studied by ROCHE.‡ There is a critical equipotential on which double points occur all round the equator, and this is the equipotential

$$\Omega = \tfrac{3}{2}\,(\mathrm{M}\omega)^{\frac{2}{3}}.$$

* R. EMDEN, 'Gaskugeln' (Leipzig, 1907), where references to the work of previous investigators will be found.

† 'British Assoc. Report,' 1883, p. 428.

‡ "Essai sur la constitution et l'origine du Système Solaire," 'Acad. de Montpellier. Section des Sciences,' VIII., p. 235. A more accessible account is to be found in POINCARÉ'S 'Leçons sur les Hypothèses Cosmogoniques,' 2nd edition, p. 15.

The equatorial radius is $M^{\frac{1}{3}}\omega^{-\frac{2}{3}}$. By a simple integration the volume of this critical equipotential is found to be

$$32\pi M\omega^{-2}\int_0^{\frac{1}{6}\pi}\frac{4\cos^2\theta-3}{4\cos^2\theta-1}\cos^2\theta\sin\theta\,d\theta = 32\pi M\omega^{-2}\times 0{\cdot}0225466.$$

This volume is of course equal to $M/\bar{\rho}$ where $\bar{\rho}$ is the mean density, from which we find

$$\omega^2/2\pi\bar{\rho} = 0{\cdot}360746. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (180)$$

Since $\bar{\rho} = 0$, the critical value of ω is also zero, and the innermost strata differ only imperceptibly from spheres. Thus, when $\gamma = 1\frac{1}{5}$, equatorial break-up occurs as soon as the mass is set into rotation at all, and therefore long before there can be any question of the pseudo-ellipsoidal form being attained.

At the other end of the scale ($\gamma = \infty$) comes the incompressible mass, for which the ellipsoidal form is attained long before there is any question of equatorial breaking up.

Thus for some value of γ, intermediate between $\gamma = 1\frac{1}{5}$ and $\gamma = \infty$, there must be a crossing over from equatorial break-up to fission through pseudo-ellipsoidal and pear-shaped figures. For a mass for which γ has this critical value, the point of bifurcation is reached and the pseudo-ellipsoidal form assumed at the very instant at which equatorial break-up is about to begin.

49. A comparison with the corresponding two-dimensional problem is of interest at this stage. Here again a solution in finite terms is only possible for one value of γ other than $\gamma = 2$, and here again this value happens to be that one for which the total mass is first finite, while the matter extends to infinity. The value in question is $\gamma = 1$ and the solution is

$$\rho = A(1+r^2/a^2)^{-2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (181)$$

which may be compared with the three-dimensional solution (179). Again it is clear that, somewhere between $\gamma = 1$ and $\gamma = \infty$, there must be a critical value of γ at which a transition occurs from equatorial break-up to fission into two detached masses. The two-dimensional problem can, however, be fully solved, and the critical value is found to be $\gamma = 2$ exactly.*

Seeing that the two problems run fairly close together in all their essential features, we might suspect that the critical value in the three-dimensional problem would not be very far from 2. An alternative guess might be $\gamma = 2\frac{2}{5}$, this corresponding more closely to the two-dimensional value $\gamma = 2$, since $\gamma = 1\frac{1}{5}$ in the three-dimensional problem has been found to correspond to $\gamma = 1$ in the two-dimensional problem.

50. In the present paper the critical value of γ has been shown to be determined in the three-dimensional problem by the equation

$$1+\epsilon[(\gamma-2)-1{\cdot}0509]+\epsilon^2[\tfrac{1}{2}(\gamma-2)^2-0{\cdot}4063(\gamma-2)-0{\cdot}0510]+\ldots = 0, \quad . \quad (182)$$

* 'Phil. Trans.,' A, vol. 213, p. 471.

in which ϵ must of course be put equal to unity for a gas in which $\sigma = 0$ at the boundary. The series on the right is probably rapidly convergent, but sufficient terms have not been calculated for the value of γ to be determined with accuracy. Neglecting all terms beyond those written down, the root of equation (182), when $\epsilon = 1$, is found to be $\gamma = 2{\cdot}1521$, but the remaining terms appear likely to increase this value somewhat, and we may perhaps take $\gamma = 2{\cdot}2$ as an approximate value. This, we may notice, is just half-way between the two guess-values considered in § 50.

When the density at the boundary is not zero, the critical value of γ is less than this; for instance, a critical value $\gamma = 1$ corresponds to a value of ϵ equal to about three-quarters.

An approximately accurate drawing of the critical figure, when $\epsilon = 1$, is shown in fig. 4, the inner curves being strata of constant density.

51. We have considered the effect of heterogeneity in the structure of the matter, and have found that a sinking of the heavier elements to the centre of the mass will result in an increase in the critical value of γ. There is no limit to the amount of

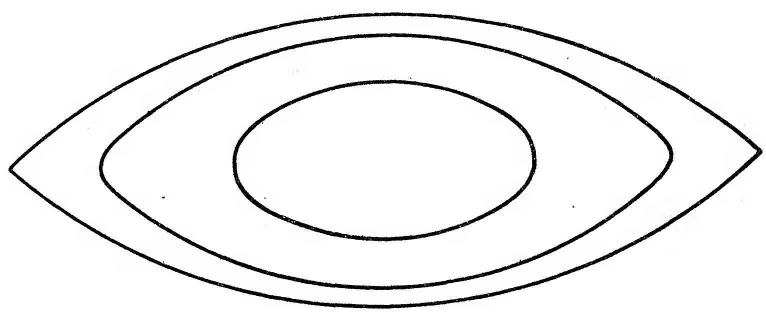

Fig. 4.

increase that can be produced in this way, although naturally the amount of increase depends on the extent to which the light and heavy elements are separated and on the ratio of their amounts. As an illustration, we considered the extreme case of a core of heavy incompressible material which we called iron, surrounded by an atmosphere of much lighter material which we called hydrogen, the two elements being supposed to be completely separated. If the volume of hydrogen was initially greater than about one-third of that of iron, the composite mass set into rotation will first disintegrate through equatorial loss of matter.

A drawing of the critical figure has already been shown on fig. 3 (p. 200); this may be compared with fig. 4.

52. There is, however, a very essential difference between the failure of a uniform mass to attain the ellipsoidal form and the corresponding failure of a heterogeneous mass. Briefly speaking the former is permanent, while the latter is transitory.

Consider for simplicity a mass of perfectly uniform gas—say helium for which $\gamma = 1\frac{2}{3}$—set into rotation and continually shrinking by cooling. The value of γ is

certainly below the critical value (about 2·2), so that as the rotation increases matter will be thrown off from the equator before the ellipsoidal shape can be reached. No matter how much material is ejected in this way, the central mass remains a mass of helium with $\gamma = 1\frac{2}{3}$, and so can never attain the ellipsoidal form. The mass will disintegrate completely by loss of matter from its equator, and there can be no fission into separate masses—unless, of course, the helium ultimately so changes its character that the average value of γ becomes greater than about 2·2.

Contrast this with the behaviour of the mass of iron and hydrogen already considered. The mass will lose matter by equatorial ejection, but the matter lost will consist entirely of hydrogen, and so this process will continually diminish the ratio of hydrogen to iron, and as the process continues the mass will continually approximate to a mass of incompressible iron. As soon as the ratio of hydrogen to iron is reduced to about one-third, the pseudo-spheroidal form becomes unstable, and the mass will assume a pseudo-ellipsoidal form.

53. We are now in a position to follow the changes in a mass of gas whose rotation continually increases through shrinkage. At first we may assume the gas to obey the ideal laws, so that γ will be less than $1\frac{2}{3}$, and, as the rotation increases, a stage will be reached at which matter is ejected from the equator. Some of this matter will perhaps fall back on to the rotating mass, but some must also pass to infinity, this latter representing a real loss of mass and of angular momentum to the rotating body. The loss must be at such a rate that the figure of the rotating body always remains a pseudo-spheroid with a sharp edge, and the velocity of rotation remains always exactly equal to the critical velocity corresponding to this critical figure.

When γ is equal to $1\frac{1}{5}$, the critical angular velocity is given by equation (180), so that $\omega^2 = 0{\cdot}36 \times 2\pi\bar{\rho}$. For the greatest value of γ for which the pseudo-spheroidal figure is possible (about $\gamma = 2{\cdot}2$), the value of the critical angular velocity is given by equation (178), and this seems to be converging to a value not far from $\omega^2 = 0{\cdot}36 \times 2\pi\bar{\rho}$. We may perhaps conjecture that for all masses of gas which are throwing off matter equatorially the value of ω^2 is nearly equal to $0{\cdot}36 \times 2\pi\bar{\rho}$.

As the mass shrinks $\bar{\rho}$ becomes greater, so that rotation becomes more and more rapid. A stage is reached in time in which the ideal gas laws no longer hold, owing to the distance apart of the molecules having become comparable with their diameters. The value of γ now increases beyond its value for an ideal gas, and values of γ greater than $1\frac{2}{3}$ become possible. When a value of γ is reached which is about equal to 2·2 if the mass of gas is perfectly mixed, but may be greater if the mixing is imperfect, the pseudo-spheroidal form becomes unstable, the mass assumes the pseudo-ellipsoidal form and the process probably ends by fission into two detached masses.

54. The value $\gamma = 2{\cdot}21$ has been found by KOCH for air at a pressure of 100 atmospheres and a temperature of $-79^\circ{\cdot}3$ C., the corresponding density being 0·23. Partly from this observational material, and partly from general theoretical principles, we may anticipate that the value $\gamma = 2{\cdot}2$ will be attained at a density

of about one-quarter of that of the substance in its solid state, and for the permanent gases this may be taken to be about a quarter of the density of water.

Thus a mass of gas will lose matter equatorially until it has shrunk to a density of about a quarter of that of water, after which it will elongate and divide up. Assuming the relation $\omega^2 = 0{\cdot}36 \times 2\pi\bar{\rho}$, the mean density $\bar{\rho}$ and the period of rotation in days (P), will be connected, so long as the mass is losing matter equatorially, by the relation

$$\bar{\rho} = 0{\cdot}035 \div \mathrm{P}^2,$$

the density $\bar{\rho} = \frac{1}{4}$ corresponding to a period of about 9 hours and $\bar{\rho} = \frac{1}{2}$ to a period of $6\frac{1}{3}$ hours. When this stage is reached the process of elongation followed by fission begins. Assuming that when fission is complete we have two stars of approximately equal mass and mean densities $\bar{\rho} = \frac{1}{2}$ revolving round one another almost in contact, the period of this system would be about a day.

55. The critical density which we have conjectured to be about one quarter is perhaps not far from that of the average B-type star.* Thus, subject to the assumptions on which we have been working, fission ought to begin at about B-type. This is in very close agreement with the results obtained by CAMPBELL in his "Second Catalogue of Spectroscopic Binary Stars."†

It is not, however, in agreement with the results obtained by SHAPLEY ‡ in his "Study of the Orbits of Eclipsing Binaries." Adopting RUSSELL's view of stellar evolution, our result would show that giant stars (except, possibly, of A-type) should be pseudo-spheroids; only B and dwarf stars could form binaries. SHAPLEY discusses 93 systems; 88 are of B or dwarf type, but only 21 have densities greater than 0·316, and only 57 have densities greater than 0·1. For one star, W Crucis, of which the orbit has been determined by RUSSELL,§ the density of the brighter component appears to be of the order of 0·000002. We are led to inquire under what physical conditions, different from those we have assumed, it can be possible for fission to occur while the density is still far below the value to which our analysis has led.

56. Consider the simplest problem of a spherical mass at rest, the equations of equilibrium being

$$\frac{\partial p}{\partial r} = \rho\frac{\partial \mathrm{V}}{\partial r} = \frac{4\pi \mathrm{G}\rho}{r^2}\int_0^r \rho r^2\, dr,$$

where G is the gravitation constant. If p_0 is the pressure at the centre and R the radius, we obtain, on integrating from the boundary to the centre,

$$p_0 = 4\pi\mathrm{G}\int_0^{\mathrm{R}} \frac{\rho}{r^2}\int_0^r \rho r^2\, dr\, dr.$$

* RUSSELL estimates the density of a giant A-type star at 0·1 ('Nature,' 113, p. 282).

† 'Lick Observatory Bulletin,' 1910.

‡ 'Princeton Observatory Contributions, No. 3' (1915).

§ 'Astrophysical Journal,' 36, p. 146.

If for any reason the ratio of increase of pressure to density is about that corresponding to the critical value $\gamma = 2{\cdot}2$, the variation of density, except near the surface, will not be very great. An approximation which will be accurate as regards order of magnitude at least will be

$$p_0 = 4\pi G\bar{\rho}^2 \int_0^R \frac{1}{r^2} \int_0^r r^2\,dr\,dr = \frac{2\pi}{3} G\bar{\rho}^2 R^2,$$

or, since $\frac{4}{3}\pi\bar{\rho}R^3 = M$ (the mass of the body),

$$p_0 = \tfrac{1}{2}G\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} M^{\frac{2}{3}}\bar{\rho}^{\frac{4}{3}} = 5\times 10^{-8} M^{\frac{2}{3}}\bar{\rho}^{\frac{4}{3}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (183)$$

Assuming the gas law

$$p = \frac{R}{m} T\rho B, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (184)$$

where R is the gas constant, and B a multiplying factor introduced by deviations from BOYLE'S law, we find for the temperature at the centre

$$T_0 = \frac{m}{RB}\frac{p_0}{\rho_0} = 5\times 10^{-8}\frac{m}{RB} M^{\frac{2}{3}}\bar{\rho}^{\frac{1}{3}}(\bar{\rho}/\rho_0),$$

in which $\bar{\rho}/\rho_0$ may, in the case of equilibrium with γ equal to about $2{\cdot}2$, be put equal to about $0{\cdot}6$, giving

$$T_0 = 3\times 10^{-8}\frac{m}{RB} M^{\frac{2}{3}}\bar{\rho}^{\frac{1}{3}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (185)$$

The energy of radiation at the centre is σT_0^4 per unit volume, where σ is STEFAN'S constant of which KURLBAUM'S value is $7{\cdot}06\times 10^{-15}$, and the pressure of radiation, being one-third of this, is $\frac{1}{3}\sigma T_0^4$ in all directions.

Denote the pressure of radiation by p_R and the gas pressure by p_G. If p_0, the pressure at the centre may be regarded as arising mainly from gas pressure, we have

$$\frac{p_R}{p_G} = \frac{\frac{1}{3}\sigma T_0^4}{p_0} = 4\times 10^{-38}\left(\frac{m}{RB}\right)^4 M^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (186)$$

which is independent of $\bar{\rho}$.

For a gas of molecular or atomic weight 32 the value of m/R is about 4×10^{-7}, while the value of B will not differ greatly from unity until very high densities are reached. Thus equation (186) becomes approximately

$$\frac{p_R}{p_G} = 10^{-63} M^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (187)$$

It will only be when this ratio is small that the radiation pressure will be negligible in comparison with gas pressure at the centre of the star, and this fixes a limit to M of the order of 10^{31}, independently of the density of the star.

Most stars whose mass is known* have a mass comparable with that of our sun ($M = 2 \times 10^{33}$), and for these the ratio (187) is far from negligible. Thus it appears that for stars of mass comparable with our sun, and of molecular or atomic weight about 32, the pressure of radiation will not be negligible in comparison with gas pressure ;† the equations of equilibrium must be replaced by

$$\frac{\partial}{\partial r}(p_G + \tfrac{1}{3}\sigma T^4) = \rho \frac{\partial V}{\partial r},$$

and our calculation fails from equation (184) onwards.

57. If we had taken a molecular weight 2, instead of 32, the value of p_R/p_G would have been reduced by a factor $(16)^{-4}$, and we should have had approximately

$$\frac{p_R}{p_G} = 10^{-68} M^2,$$

a ratio which may be considered small when M is of the order of 10^{33}. Now whether we consider that radiation pressure is comparable with gas pressure or not, the temperature at the centre of stars such as we have considered is of the order of 10^7 degrees Centigrade, and at such temperatures it seems probable that matter would to a large extent be broken up into its constituent electrons and nuclei. For purposes of calculation of gas pressure each electron behaves like the molecule of a gas, and the effect of electronic disintegration is to reduce the effective molecular weight. It is readily seen that when electronic disintegration is complete, a limiting effective molecular weight of 2 is reached for all substances except hydrogen.‡

Further, radiation pressure when it is appreciable may be treated as arising from molecules of molecular weight zero, and so the effective molecular weight may be still further decreased.

* The stars whose masses are known are bright binaries—binaries because there is no means of determining mass except by the mutual action of two bodies on one another, and bright because the fainter binaries escape observation. The conclusion of this paper is that bright binaries are binaries of large mass, the mass being of order of magnitude greater than 10^{31} gm. This is borne out by the observed masses of those binaries which are bright enough to have attracted attention, but there is nothing to show that there are not a great number of less bright binaries of smaller mass. It rather appears as if the few well-determined masses of binaries are not likely to give a good sample of the masses of all stars. It is perhaps significant that RUSSELL'S well-known diagram of absolute magnitudes ('Nature,' vol. 113, p. 252) shows a range of something like five magnitudes (ratio 100 to 1) for dwarf stars of similar spectral type, suggesting a range of masses enormously greater than that calculated from observations on binary stars.

† This agrees with the result stated by EDDINGTON ('M.N., Royal Ast. Soc.,' vol. 77, p. 16), but as the question is of some importance for our present investigation, I have thought it worth giving a separate discussion freed from the special assumptions of EDDINGTON'S paper. EDDINGTON'S statement that radiation pressure is practically negligible for dwarf stars does not appear to be altogether confirmed by my equation (187).

‡ *Cf.* EDDINGTON, 'M.N., Royal Ast. Soc.,' vol. 77, p. 596.

Thus the effective molecular weight, regarded as being determined by equation (184), may fall to very small values as we pass to the interior of a star. In § 43 we considered the tendency for greater molecular weights to occur in the interior of the star, and found that it would delay the stage at which the pseudo-spheroidal form would become unstable. It now appears that, in order to represent actual conditions, we should have examined the reverse tendency. Effectively lighter molecular weights may be expected to occur in the interior of a star, and this will cause the process of fission to begin at lower densities. In a general way we may expect that stars of greatest mass will begin the processes of fission in the earliest stages of their careers. If so, such a star as W Crucis, assuming that its two components have been formed out of the fission of a single body, must be a star of very great mass indeed.

58. To sum up, we have found that a star of small mass (say $\frac{1}{10}$th that of our sun), in which there is no great amount of atomic disintegration, and in which pressure of radiation does not play a prominent part dynamically, will not begin to break up into a binary star until it reaches a density of from one-quarter to one-half that of water. In more massive stars there will be considerable atomic disintegration, and pressure of radiation will be dynamically important. Such stars will break up at lower densities than smaller stars.